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High frequency pricing of Asian options

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Abstract

In this thesis, we analyse an efficient way to price Asian options using a high-frequency perspective. First, we review the theoretical concepts and the literature about high frequency trading, options and option pricing models, specifically focusing on Asian options. Second, this thesis' empirical contribution is to replicate Hsu, Lin, and Kuo (2020)'s paper. Also, we incorporate the study of Lin and Chang (2020). In both these 2020 papers, the authors extend Hull and White's model using a Taylor series expansion, which allows them to derive an approximate analytical solution for Asian options under stochastic volatility conditions. In comparison to the Monte Carlo simulation, we examine this method's computational efficiency and its ability to deliver sufficiently accurate prices. Finally, we discuss both the accuracy and the computing time of this approach which illustrates that the model could be sufficient for handling options in the context of high frequency trading.

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"Je soussigné, RUCHAT, Baptiste, Master en gestion de l'entreprise (double diplôme), déclare par la présente que le Mémoire ci-joint est exempt de tout plagiat et respecte en tous points le règlement des études en matière d'emprunts, de citations et d'exploitation de sources diverses signé lors de mon inscription à l'ICHEC, ainsi que les instructions et consignes concernant le référencement dans le texte respectant la norme APA, la bibliographie respectant la norme APA, etc. mises à ma disposition sur Moodle.

Sur l'honneur, je certifie avoir pris connaissance des documents précités et je confirme que le Mémoire présenté est original et exempt de tout emprunt à un tiers non-cité correctement."

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General Introduction

When it comes to trading, one still might think about someone yelling on the phone in a trading room and flailing his arms around, like in the renowned Martin Scorsese's movie: The Wolf of Wall Street. But times have changed. Computers and mathematics have now the upper hand in most financial transactions. In recent years actually, high frequency trading (HFT) has entered the global financial markets. Financiers and trading firms then engaged in a ruthless battle for speed. From the stock market's early days to the democratisation of trading, through the advent of electronic trading in the 1990s and the growing development of HFT, technological breakthroughs were made possible by the development of complex mathematical and algorithmic methods. In that context, we carried out this master thesis. In a high-frequency perspective, we then sought to develop an average option pricing tool. The purpose of this tool is to enable high frequency traders (HFTs) to do their job, given the requirements in terms of execution speed. We therefore seek to answer the following research question: how to price Asian options in the context of HFT?

To produce this master thesis, the following methodology has been applied. First, we reviewed the theoretical concepts and the literature about HFT, options and option pricing models. To do this, we mainly used Google Scholar to access relevant academic articles on the presented subjects. Besides, we used multiple official reports. The literature review therefore provides a reliable source of information for exploring the aforementioned aspects. Second, we used the Python programming language to apply the gathered theory. This has been done using the Anaconda Navigator for coding in Jupyter Notebook and Spyder.

My interest in HFT and technologies surrounding the financial world has been developed since my undergraduate studies. In addition to that, my attraction to computer programming has also been sparked in recent years as I strongly believe it can be closely related to the financial field. During my internship as an investment banking intern, I concretely realised that the automation of repetitive tasks was made possible by programming and coding effective tools and software. The latter significantly improve teams' productivity and efficiency. To this end, I started learning to code. Through this master thesis, I wanted to improve my skills in linking computer programming to finance. I am confident that this will be a valuable asset for my future career.

To set the context for the above question, we first review the existing literature regarding HFT. Throughout this first chapter, we define what HFT is and we address its characteristics, as depicted by the U.S regulatory instance. Also, we discuss the regulation. In the end, we examine different HFT strategies.

In the second chapter, we define the elements of the research question in more detail. To do so, we investigate the functioning of options as well as various option styles. In order to ensure consistency with the proposed research question, we reasonably focus on Asian options in this chapter. Again, we rely on academic papers from the existing literature.

The third chapter is devoted to option pricing models. We first bring some theoretical elements to the concepts put forward. Among other things, we highlight the Monte Carlo simulation which has been used as a benchmark in the fourth chapter. We conclude this chapter with some concrete numerical applications of the models studied.

After highlighting the context of the research question and its theoretical elements, we turn to the fourth chapter on high frequency pricing of Asian options. In that chapter, we expose the approximate analytic solutions proposed by Hsu et al. (2020) and Lin and Chang (2020) regarding Asian options. Next, we compare the accuracy and efficiency of both solutions with the benchmark of Monte Carlo solutions we programmed.

Ultimately, we conclude this master thesis by providing answers to the above-mentioned research question. We also discuss the encountered limitations and other possible avenues to explore for a future research.

Chapter 1

High Frequency Trading

In this first chapter, we set out the context in which our research question has been raised. First, we define HFT. We will see that this recent technique derives from older elementary methods, which we also study in Section 1.1. In Section 1.2, we discuss the flash crash of May 2010 which brought HFT to general public and regulatory attention. This leads to a discussion of regulations in more detail in Section 1.3. Lastly, we study several HFT strategies in Section 1.4.

1.1 Definition of high frequency trading

In recent years, HFT has gradually gained a foothold in financial markets. This development has been made possible and encouraged by the interaction of legislative measures, increased competition between execution venues and significant advances in information technology (Chlistalla, 2011).

Although the U.S. Securities and Exchange Commission (SEC) itself recognises that the term is rather recent and that there is no clear definition of HFT, many authors have tried to define this relatively modern practice. For example, Kearns et al. (2010) define HFT as the ability of traders and algorithms to place and therefore execute or cancel orders in an extremely short time frame, from as little as 10 milliseconds up to 10 seconds. Another definition given by Hagströmer and Norden (2013) specifies that HFT firms are primarily involved in proprietary trading only and use algorithms in their trading strategies, that is, the HFTs do not work on behalf of clients. Nevertheless, in a Concept Release on Equity Market Structure, the Securities and Exchange Commission (2010a, p.45) proposes a slightly broader definition, compared to Kearns et al. (2010), in which HFTs are "*professional traders acting in a proprietary capacity that engage in strategies that generate a large number of trades on a daily basis*". This definition includes in particular the idea later put forward by Hagströmer and Norden (2013) pertaining to the proprietary trading aspect. In addition to that definition, the Securities and Exchange Commission (2010a, p.45) attributes to HFTs five characteristics, namely: "*(1) the use of extraordinarily high-speed and sophisticated computer programs for generating, routing, and executing orders; (2) use of co-location services and individual data feeds offered by exchanges and others to minimize network and other types of latencies; (3) very short time-frames* *for establishing and liquidating positions; (4) the submission of numerous orders that are cancelled shortly after submission; and (5) ending the trading day in as close to a flat position as possible (that is, not carrying significant, unhedged positions over-night)*". However, some of those characteristics highlighted by the SEC seem to be somewhat contested. For instance, not all market participants agree with the fifth characteristic. Jones (2013) suggests that many HFTs do hold significant inventory positions overnight. Nevertheless, the definition provided by the SEC seems to be a workable definition of HFT for most market players (Jones, 2013). Harris (2013, p.6) also complements previous definitions by underlining that "*HFT is trading for which success depends critically on low-latency communications and decision making*".

Looking back at the definition issued by the Securities and Exchange Commission (2010a), it is not straightforward to determine accurate statistics on HFT. This difficulty is mainly explained by the extreme confidentiality of information that the various market players wish to preserve. Nonetheless, it is estimated that the share of transactions involving HFTs in the U.S. market has soared from about 20% in 2005 to roughly 70% today. In Europe however HFT market share is estimated to be lower at around 30% in 2010 and 50% of all trading transactions today (Autoriteit Financiële Markten, 2010; Petitjean, 2021). However, according to the Autoriteit Financiële Markten (2010), a lack of clear and unanimous definition of HFT makes classification difficult. This situation leads for example to other problems such as inaccurate estimates of HFT market share and the incapacity to precisely estimate the reach and influence of HFT in its markets (Zaharudin et al., 2022).

In order for us to understand HFT and its characteristics as depicted by the SEC more fully, we will now explore each of them in turn:

"(1) the use of extraordinarily high-speed and sophisticated computer programs for generating, routing, and executing orders".

Although HFT has brought new efficiency to global trading, it has also started an unrelenting competition for speed, leading to a meticulous, subterranean battle among HFT algorithms. In today's HFT, time is measured in nanoseconds (billionths of a second, or 10^{-9} second), compared to microseconds (millionths of a second, or 10^{-3} second) some years ago (Aquilina et al., 2021). This represents the fastest possible signal for an order or a trading instruction to travel some distance (Petitjean, 2021). Within a nanosecond, light in a vacuum travels a distance of about 30 centimetres. At such high speeds, having a cable that is shorter by as little as a meter thus saves three to five nanoseconds. For that reason, HFT is exquisitely sensitive to the length and transmission capacity of the cables connecting computer servers to exchanges systems. Also, the speed of transmission of signals among various data centres is a crucial factor as HFT needs to be as quick as possible (MacKenzie, 2021).

In order to satisfy this ever-increasing need for speed, HFT firms are using fibre-optic cables that span the globe across continents, seas and oceans (Lange, 2017). In the early years of HFT, transmissions between data centres were generally via laser-generated pulses of light in fibre-optic cables (Hautsch, 2017). However, the materials from which cable strands were made slowed those light pulses. This method consequently did not provide a sufficient and optimal speed. Another solution was to use a wireless signal sent through the Earth's atmosphere. Again, such a solution does not deliver optimal results. Because wireless transmission for HFT necessitates radio frequencies that are in high demand, tailor-made radios, and antennas in particular locations, it is much more expensive than the routine use of fibre-optic cable usually is (MacKenzie, 2021).

One way of assessing the speed of HFT is the response time of an HFT firm's system, i.e., the time delay between the arrival of a signal (a data pattern that informs an trading algorithm) and an action (e.g.: sending or cancelling an order) in response to that signal. For illustration purposes, a technical presentation by the futures exchange Eurex in September 2018 stated that the European Exchange had recorded response time as fast as 84 nanoseconds. This is currently the fastest response time relative to HFT that has been recorded quasi-publicly (MacKenzie, 2021).

"(2) use of co-location services and individual data feeds offered by exchanges and others to minimize network and other types of latencies".

As we have seen previously, proximity to markets matters in HFT. The first traders to see market movements are those closest to the market. They thus have a noteworthy advantage compared to competitors (Brogaard et al., 2015). With the race for speed between HFT firms, new practices emerged. When nanoseconds became important, HFT businesses were prepared to spend a fortune to be as near as possible to execution venues and exchanges, aiming at time saving. For those HFT firms' needs, trading firm's co-located servers have been created, sometimes even within exchanges. Because servers' locations are crucial to speed, companies (most of the time the exchanges themselves) that own or control key locations and means of transmission can charge heavily for access to them (MacKenzie, 2021). Colocation services are real additional sources of revenue to exchanges and reduce latency for users as well (Brogaard et al., 2015). Colocation is defined by MacKenzie (2021, p.165) as "*the placement of trading firms' servers in the same building as an exchange's computer system*", referred to as a data centre. The aim is simply to get as close as possible to the servers of the market company. By way of comparison, this is no more and no less than the resumption of a traditional practice that allowed brokers to pay more for space near the pit (Daniel, 2012). However, most major exchanges have now established a policy that requires a fixed and equal length for the cables that connect each co-located trading firm's servers to the exchange's system. This policy requires, among other things, that too short cables be lengthened because every centimetre counts (MacKenzie, 2021). As Harris (2013) stated all right in his definition, HFT success depends substantially on minimizing latencies for HFT firms.

Turning now to wireless antennas, HFT firms can spend millions for a chosen place on a data centre roof. At such a competitive level, the objective is not any longer just to position antennas as close as possible to a data centre, considered simply as an entire building but rather to take into account the precise location where the cable can enter that data centre from the nearest antennas. For example, the Chicago Mercantile Exchange (CME) has never authorised firms to situate their antennas in what would be the very optimum location. Only a small set of entry points may be used, and the CME does not permit HFTs to place their equipment closer than 30 meters to any of these entry points (MacKenzie, 2021).

"(3) very short time-frames for establishing and liquidating positions".

A review of the definitions of several authors of HFT shows that most of them seem to be able to agree on one common characteristic that is the ability to place, execute and cancel orders in extremely short time frames. Nonetheless, if such an ability is applied to initiate buy or sell positions that are subsequently held for hours or even many minutes or seconds, the whole point of HFT is lost. In a sense, the profit of a given high frequency trade has to be realized almost instantly after it is executed. In which case, only HFT would be able to capture the profitability from a trade by rapidly liquidating the initiated position. The other option would then consist of holding trades over a longer time horizon. In that second case, almost any competent electronic trading platform can capture profit from it (Kearns et al., 2010). As we noticed in previous definitions, HFT strategies based on very short-term positions are set up to eliminate any risk carried between two trading sessions as HFTs usually do not carry positions over-night. We will come back to this point a little later.

Another reason for very short time-frames positions lies in a massive number of orders sent during one trading session. Since most of the HFT strategies are based on capturing infinitesimal price differentials, profitability and success of HFT strategies are based on making constant trips back and forth to the market (Daniel, 2012).

By way of illustration, the average holding period of shares listed on the New York Stock Exchange (NYSE) was two years in the 2000s. Six years later, it was estimated to be eleven months in average, but this does not take HFT into account. Considering it, researchers estimated this average holding period was 22 seconds (Gomez, 2016).

"(4) the submission of numerous orders that are cancelled shortly after submission".

Of more than one million orders placed in a second around the globe, 95% are modified of cancelled (Declerck and Lescourret, 2015). This is in fact a major characteristic of HFT in addition to speed. The ratio between the number of transactions and the number of orders submitted is consequently very low.

In their paper, Brogaard et al. (2015) studied the trading activity and performance of HFTs on OMX Stockholm 30[1], referenced in Appendix A. Among their observations, we mainly concentrate here on the number of trades and the number of cancellations for fast traders, i.e., colocated traders as represented in Brogaard et al. (2015). For that category, HFTs are also broken down into three colocation types, namely basic, premium and 10G. Such a distinction will not be taken that much into account under this fourth characteristic as it is more relevant for previous HFT characteristic regarding colocation. This segmentation between colocated traders simply represents the differences related to the quality of the colocation service (Brogaard et al., 2015).

With a focus on the overall trading activity, 36 trading entities have been observed over a period of 25 days. For those HFT firms, the trading volume amounted to 3,211.2 trades per day which represents a nominal value of $181.6 million. Looking at the fraction of total activity underscores that only 44.2% of trades are effectively executed. In addition to that, colocated traders submit

[1]OMX Stockholm 30 is the main stock market index on the Stockholm Stock Exchange in Sweden. It is managed by the OMX Group and lists the prices of the 30 companies whose shares are most heavily traded (Nasdaq, Inc., 2022c).

83% of all limit orders on the OMX Stockholm 30. Of these limit orders, an enormous percentage (85.8%) of trades are cancelled. Observing more closely the 10G colocation group only, authors find that the fastest traders are responsible for more than 60% of all cancellations even though only two HFT firms are categorized in this segment. Premium colocated traders cancelled 24% of orders, while basic colocated HFT firms are responsible for only 1.2% of all colocated traders' cancellations (Brogaard et al., 2015).

To slightly counteract these practices of cancelling huge amounts of orders, exchanges set up interesting initiatives. For instance, the Nasdaq Stock Exchange in the United States requires a minimum ratio of trades to orders placed with the aim of limiting order cancellation activity. By way of example, this ratio amounted to 1/100 in 2015. Besides, the IEX Exchange, an alternative platform also based in the United States with the goal of mitigating the effects of HFT, requires a minimum order execution time (Declerck and Lescourret, 2015).

"(5) ending the trading day in as close to a flat position as possible (that is, not carrying significant, unhedged positions over-night)".

Very low overnight inventory position for HFTs allows, among other things, to distinguished HFT firms from a non-HFT after comparing the day closing inventory position level (Hossain, 2022). Indeed, owning a financial asset between two trading sessions usually adds an additional risk to holding the asset. As we can observe in financial markets, the difference between the closing price and the next trading day opening price of that same asset may have varied more significantly following the occurrence of some events. For that reason, if HFTs aim closing the day with a near net zero position in their financial assets, they should loosen their trading up before the end of the trading day (Brogaard, 2010). Nevertheless, this characteristic seems to be less obvious to everyone as some market participants do carry large inventory positions between trading sessions (Jones, 2013). The fact that inventory is non-zero for some HFT firms may suggest, for instance, that their strategies extend over several order books which would be the case with arbitrage strategies (Hagströmer and Norden, 2013).

1.1.1 Understanding algorithms and algorithmic trading

According to the Autoriteit Financiële Markten (2010) and Petitjean (2021), HFT is in fact a subset of algorithmic trading (AT). Under Article 4(1)(39) of MiFID II, AT is defined as "*trading in financial instruments where a computer algorithm automatically determines individual parameters of orders such as whether to initiate the order, the timing, price or quantity of the order or how to manage the order after its submission, with limited or no human intervention, and does not include any system that is only used for the purpose of routing orders to one or more trading venues or for the processing of orders involving no determination of any trading parameters of for the confirmation of orders or the post-trade processing of executed transactions*" (European Securities and Markets Authority, 2021, p.11). Several years after its Concept Release on Equity Market Structure, the SEC recently acknowledged in a report on Algorithmic Trading in U.S. Capital Markets, that "*algorithmic trading, in one form or another, is an integral and permanent part of our modern capital markets*". It also adds that "*the use of algorithms in trading is pervasive in today's markets*" (Securities and Exchange Commission, 2020, pp.4-5).

To understand clearly what AT is, it seems important to first learn more about electronic trading. As of the end of the 1990s, the electronification of execution venues allowed market participants such as banks, brokers, and their clients to access electronic order books remotely. This therefore refers to the appearance of electronic trading (Chlistalla, 2011). First computers and screens were then able to display stock prices. At that time, automated trading did not represent trading performed by autonomous machines but simply the implementation of instruments allowing an automated dissemination of information and the dematerialization of order placement. In addition to that, the computerisation of trading rooms has made it possible to systematic calculations and trade-offs on certain stocks and indexes (Lebreton, 2008). Electronic trading consequently represents the ability to transmit orders electronically as opposed to via telephone, mail or in person, when it used to be. Nowadays, most orders are transmitted and communicated via computer network which involves that the term electronic trading has promptly become redundant (Chlistalla, 2011).

Considered as a kind of artificial intelligence (AI), mathematical algorithms (MAs) appeared in the field of finance once financial markets started to computerise. The use of MAs thus emerged as a consistent way of programming a stock market operation with regards to the behaviour of a variable (Hoste, 2015). An algorithm can be defined as "*a set of mathematical instructions or rules that, especially if given to a computer, will help to calculate an answer to a problem*" (Cambridge University Press, 2022). In France, the Autorité des Marchés Financiers (AMF) differentiates two types of MAs. On the one hand, trading algorithms are used to identify market opportunities and initiate buy and sell orders. On the other hand, execution algorithms allow the implementation of the order execution (Oseredczuck, 2011). In a more sophisticated way, we also find genetic algorithms (GAs). This type of algorithm is mostly applied to machine learning and problem optimization purposes. GAs are "*search based algorithms established on the concepts of natural selection and heredity. (...). In GAs, for a given problem, we have a varied and numerous solutions*" (Lambora et al., 2019, p.381). Vallée and Yildizoğlu (2004, p.14) state that GAs are "*a valid approach to many practical problems in finance which can be complex and thus require the use of an efficient and robust optimisation technique*". Among other uses, GAs are useful for trading rules employed to AT. Those algorithms are generally extremely confidential but must be accessible to programmers who will give different work instructions. MAs and GAs can be coded with different programming languages such as Python, which we use in this master thesis. Most commonly languages are Java, C# as well as C++ (Lebreton, 2008).

Algorithms such as VWAP (volume-weighted average price), TWAP (time weighted average price), TVOL (target volume), IS (implementation shortfall) and PVOL (percent of volume) all work on the same principle. Massive and large orders are broken down into smaller, more marketable orders. Those algorithms, in fact, have been around for quite some time. AT simply took over the strategies and allowed traders to run them automatically while HFT carried them out more rapidly. A VWAP transaction, for example, will be 40% in the morning and 60% in the afternoon, based on the exchange's observed trade volume (Lebreton, 2008). TWAP algorithm will undertake a series of tiny operations over a period of time. For instance, it will split a parent transaction into smaller child orders at constant time interval (Kolm and Maclin, 2010). PVOL and TVOL algorithms also split orders that are too large, this time according to the volume. On its side, IS algorithm carries out all transactions considering market conditions (Lebreton, 2008).

In AT, above-mentioned algorithms were originally developed to be used by the buy-side so as to manage orders and reduce market impact by optimising trade execution once the buy-and-sell decisions had been made elsewhere. Though, other algorithms exist and are used by many different types of market participants. Some algorithms frequently combine active and passive strategies, using both limit orders and marketable orders. For instance, hedge funds and broker-dealer are using computers to supply liquidity. Liquidity demanders often use smart order routers to determine where to send an order. Additionally, statistical arbitrage funds need algorithms to rapidly deal with huge volumes of information contained in the order flow and diverse securities' price moves. Finally, institutional investors are using various algorithms to trade large amounts of stock progressively over time. In all cases, those algorithms dynamically screen market conditions across numerous securities and stocks on many trading venues. A major asset of these algorithms is also their ability to reduce market impact by optimally and sometimes randomly breaking large orders into smaller ones, and closely tracking benchmarks over the execution interval (Hendershott et al., 2010).

AT is unquestionably a more complex version than electronic trading. It is often perceived as a generic term because it does not necessarily imply aspects such as speed, which is typically associated with HFT, as discussed sooner. With massive computerisation, execution venues propose continuous quotations and offer an ever-increasing order management capacity. Computers are powerful enough to perform calculations in real time. The thought is then to optimize decision-making treatments along with transactional costs (Lebreton, 2008). As many market participants now employ AT, Hendershott et al. (2010, p.1) provide a simpler definition by describing AT as "*the use of computer algorithms to automatically make certain trading decisions, submit orders, and manage those orders after submission*". To some extent, AT is an advanced evolution of electronic trading whose parameters are set by a rigorous adherence to programmed set of rules geared toward at producing specific execution results (Chlistalla, 2011).

According to a report by Coherent Market Insights (2019), global AT market was valued at $10.347 billion in 2018 and is expected to reach $25.257 billion in 2027. The global AT market is consequently projected to witness considerable expansion within the coming five years. Such a growth is mainly attributed to increasing adoption of cloud-based services and solutions together with cloud computing for AT as professional cloud services market is expected to reach $41.59 billion by 2023. In addition to that, upward demand for market surveillance by traders and increasing need for AI-based services are considered as key factors of driving growth of the AT market. AI actually helps to adopt market conditions, learn from past experiences and make trade decisions accordingly.

1.2 Flash crash of May 6, 2010

On May 6, 2010, financial markets in the United States were jolted by a brief but significant price decline that occurred within minutes. Because of the suddenness of the event, the market drop was eventually nicknamed a flash crash (Petitjean, 2021).

Around 2:30 p.m. on that day, the S&P 500 volatility index rose by 22.5% from its opening level. This was caused by unfavourable market sentiment arising mostly from the European debt crisis. This then prompted selling pressure among investors, which brought the Dow Jones Industrial Average (ticker symbol: DJI) down by 2.5% (Securities and Exchange Commission, 2010b). Concurrently, in order to hedge an existing equity position, a major fundamental trader, Waddell & Reed, used a sell order algorithm to sell 75,000 E-mini contracts on the S&P 500 futures at 2:32 p.m. (Kirilenko et al., 2017). HFTs and other futures market participants first absorbed the selling pressure, followed by fundamental purchasers and cross-market arbitrageurs (Zaharudin et al., 2022).

HFTs then amassed a net long position in E-mini contracts as a result of the Waddell & Reed sell algorithm order, prompting them to aggressively sell the contracts they possessed to lower their inventories. They exchanged almost 140,000 E-mini contracts on that day, which represents more than one third of the total trading volume. Later, the substantial growth in trading volume boosted market volatility, scaring away long-term traders. Due to a lack of demand, HFT firms bought and sold from one other, leading to a “hot potato” volume effect that pushed the price of the E-mini down by around 3%, while cross-market arbitrageurs simultaneously sold comparable quantities in the equities markets. This brought the price of the S&P 500 SPDR (ticker symbol: SPY) down by almost 3% as well (Zaharudin et al., 2022). All in all, over 20,000 trades were executed across 300 different securities and exchange-traded funds (ETFs) traded at prices 60% or more from their 2:40 p.m. values. By 3:08 p.m., most stock prices had fortunately returned to their rational values (Kirilenko et al., 2017).

1.3 Regulations

Since that event, HFT has received considerable public and regulatory attention. Even though a subsequent Commodity Futures Trading Commission (CFTC) and SEC investigation cleared HFTs of directly causing the flash crash, what could have been observed on that day were the effects of the violation of financial markets and the interplay between regulation, competition, and technology (Chlistalla, 2011; Kirilenko et al., 2017).

1.3.1 American regulation

In the United States, regulatory agencies spearheading the campaign against HFT are the SEC, the Financial Industry Regulatory Authority (FINRA) and the CFTC. Over the last few years, these institutions have made moves to bring HFT under greater examination (Woodward, 2017).

In 2014, the SEC approved the Regulation Systems Compliance and Integrity (Reg SCI). This regulation aspires to reinforce the technology infrastructure of U.S. securities markets. Reg SCI applies to SCI entities (Woodward, 2017). The latter are defined as SCI self-regulatory organizations, SCI alternative trading systems (ATSs), plan processors, or exempt clearing agencies subject to ARP (Securities and Exchange Commission, 2015). With a focus on HFT, Reg SCI mandates SCI entities to take part in business continuity and disaster recovery plan testing. This specification takes into account the risk of flash crashes, among other factors. Besides, in view of all market players' growing use of automated technology, the SEC has already signalled that it may broaden the scope of Reg SCI (Woodward, 2017).

Moreover, many HFT firms were exempted from FINRA registration prior to 2015. Those used to trade on exchanges through a third-party broker-dealer or on alternative trading platforms. Since then, formerly exempted HFT firms are required to register with the self-regulatory organization (Hautsch, 2017). Now, only around 125 entities are free from FINRA registration requirements under the narrower exemption. Conversely, HFTs who register with FINRA are now subject to disclosure requirements and conduct audits. This has been implemented under the careful eye of the SEC to improve regulatory surveillance of active proprietary trading firms. In this regard, a SEC commissioner declared: "*This will ensure that these [HFT firms] can be held responsible for any potential misconduct*" (Woodward, 2017, pp.28-29).

1.3.2 European regulation

Securities trading in all European Union (EU) member states is regulated by the European Securities and Markets Authority (ESMA). The independent EU regulatory agency analyses possible threats to financial stability and embrace emergency actions in times of crisis. The Market in Financial Instruments Directive II (MiFID II) and the Markets in Financial Instruments Regulation (MiFIR) currently aim to supervise organizations and trading venues that participate in AT and HFT (Woodward, 2017).

Since 2007, the MiFID regulatory package, including MiFID I, has been a cornerstone of EU financial market regulation. This aims to promote market competitiveness by creating a unified investment market and ensuring uniform investor protection. To this purpose, MiFID I developed a set of regulations and standards to prevent market abuse, transaction transparency, and instrument admission into trading. ESMA published a HFT questionnaire that polled trading businesses on their strategies, market access, latency needs, algorithm development, and risk management in order to guide future regulation efforts. The European Commission wants to revise laws based on input from these surveys. The objective is therefore to adjust the existing regulation framework to account for development in HFT and to prevent or mitigate the impact of flash crashes (Woodward, 2017). MiFID II, which came into force in January 2018, reflects a continuing attempt to reduce the negative consequences of HFT (Hautsch, 2017). In the aftermath of the 2008 financial crisis, ESMA plans for MiFID II to increase the efficiency, resilience, and transparency of financial markets. Besides, ESMA is working on a few proposed regulatory technical standards (RTS) and draft implementing technical standards (ITS) for MiFID II, with the goal of broadening the scope of regulation. For instance, new regulations will apply to a wider range of non-equity products and over-the-counter (OTC) trading (Woodward, 2017).

The April 2016 Delegated Regulation supplementing MiFID II especially targets AT and HFT by establishing standards for what counts as HFT that are comparable to those approved by the SEC and CFTC in the United States. As an example, HFT firms must inform regulators if they are engaging in HFT and on which trading venues they are trading (Woodward, 2017).

1.4 High frequency trading strategies

HFT is not a strategy as such. It is rather an advanced technology that has been developed with the computerisation and automation of financial markets. Over time, algorithms have unceasingly evolved. For example, first generation of algorithms were pure trade execution algorithms with quite simple goal and logic. More advanced algorithms such as strategy implementation algorithms then appeared and became much more complex, sophisticated. From now on, GAs incorporate intelligent logic which learns from financial market activity and adapts the trading strategy based on what is perceived on the markets (Chlistalla, 2011). HFTs consequently do no follow one single trading strategy as there now exist many different HFT strategies producing their own signals. In its 2020 Staff Report on Algorithmic Trading in U.S. Capital Markets, the Securities and Exchange Commission (2020) depicted four broad types of short-term HFT strategies: passive market-making, arbitrage strategies, structural strategies, and directional strategies. Among those strategies, it is possible to distinguish between non-directional strategies and directional strategies. Non-directional strategies logically include informal passive market-making and arbitrage strategies. Note however that structural strategies as depicted by the SEC are included under "structural arbitrage" in this thesis, which seems more relevant.

Besides, note that HFT generally does not build any new strategy. Well-known strategies and market principals are rather transforming into the HFT form (Tichánek, 2020).

1.4.1 Passive market-making

Passive market-making involves market-makers simultaneously submitting limit orders on buy (bid) and sell (ask) sides of the electronic limit order book. By doing so, HFTs provide liquidity to markets participants who want to trade immediately with market orders. Within this HFT strategy, market-makers aim to buy at the bid price and sell at the ask price. Profits are largely made by earning the spread between both sides, referred to as the bid-ask spread. Additionally, HFT firms receive discounts, called liquidity rebates, and sometimes referred to as market fees. Those are offered by several exchanges for providing resting liquidity. This allows HFT firms to generate additional revenues. The risk of such a strategy is that prices will move fast in one direction or another against their bids or asks. Passive market-makers are thus exposed to adverse selection, which can make it challenging to profitably trade out of a position (Securities and Exchange Commission, 2020). This market-making strategy would account for 65% to 71% of all HFT activity (Tichánek, 2020).

High frequency market-makers also go out on a limb that they trade with a well-informed counterparty, which could make HFT firms lose money. The latter shall thus be driven to ensure that their limit orders encompass as much current information as possible at the earliest opportunity. HFT round-trip gains are smaller than the ones of better-informed entities, but their global profitability originates in a greater number of round-trips (Mandes, 2016). The use of limit orders within passive market-making allows HFTs to frequently update their outstanding quotes as market conditions change and in response to other order submissions or cancellations. Furthermore, HFT market-makers might be able to refresh their orders in reaction to a price move in a correlated ETF or futures contract (Jones, 2013). Because of this continuous updating process, HFTs tend to produce massive amounts of modification and cancellation messages as mentioned before in Section 1.1.

Jones (2013, p.6) also declared that "*some HFT market-makers formally register as such with trading venues. Others act as informal market-makers*". The willingness to officially register as HFT market-maker is based on the obligations and benefits related to being a registered market-maker. Those obligations and benefits may vary depending on the assets and trading venues (Jones, 2013).

Mandes (2016) contributes to the literature by asserting that passive market-making is often referred to as quasi market-making. HFT firms can actually interrupt their activity whenever the market state would lead to generating unprofitability because they face no legal duty to maintain quotes and guarantee liquidity in the markets. Moreover, he claims that "*liquidity provided by HFT is often illusory*" as HFTs only provide liquidity in the market when they are confident to profit (Mandes, 2016, p.12). However, as we have noted, some HFT firms also try to generate extra profits by collecting liquidity rebates paid by trading venues.

In any case, human market-makers are increasingly replaced by HFT market-makers. In doing so, transaction costs for both retail and institutional traders have declined significantly (Harris, 2013).

1.4.2 Arbitrage strategies

In arbitrage strategies, HFTs typically seek to capture tiny price differences between related products or markets (Securities and Exchange Commission, 2020). There are several arbitrage strategies, which we report below.

Index arbitrage

A classic example is index arbitrage, which can take place between the index product and the individual stocks composing the index. For instance, if the S&P 500 futures price increases without the component stocks moving in price, an arbitrage strategy can be implemented. HFTs will instantly buy shares of the underlying stocks in the correct quantities until the relative mispricing is eliminated (Jones, 2013).

Statistical arbitrage

In a similar way, the S&P 500 futures price could move up due to the arrival of buy orders, but the related SPY ETF price may not go up at the same time. As both financial instruments are very similar, their price should normally move in lockstep one-for-one. In that case, HFT firms would swiftly buy SPY and trade down S&P 500 futures contracts concurrently. Due to this action, HFTs would lock in a small profit on the price discrepancies between the two instruments (Jones, 2013). It seems important to mention again that only HFT can take advantage of such small price differences in an exceedingly short period of time.

Note that the S&P 500 futures is traded in Chicago on the CME while SPY is traded on nearly every equity trading venue in the United States as well as several foreign trading venues. Given this market difference, HFT firms require prompt computer processing capability along with the most efficient possible link between Chicago's electronic market and the electronic equity markets to benefit from such arbitrage opportunities (Jones, 2013).

This example exhibits the first characteristic described by the SEC regarding HFT. The latter discloses that HFT firms need astonishingly high-speed and complex computer programs to be profitable. Because of that, any high-frequency arbitrageur faster than other market participants will be able to purchase all relatively mispriced SPY shares and sell likewise mispriced S&P 500 futures contracts. Instruments' prices will get back into line, allowing the winner to take it all. In a contrasting manner, other traders will not be fast enough and will not be able to benefit from appealing arbitrage opportunities (Jones, 2013).

Structural arbitrage

In addition to HFT arbitrage strategies described just above, Jones (2013, p.8) appends that "*relative value trading can also take place between individual securities*". An example of the Belgian company Anheuser-Busch InBev[2] (ticker symbol: ABI), which is a main component of the BEL 20 index[3], illustrates this point. This asset usually trades in Belgium but has an American Depositary Receipt[4] (ADR) that trades on the NYSE (Anheuser-Busch InBev, 2021). As with this company, some other listed corporations have multiple classes of common stock, or other equity-linked securities. HFTs can therefore turn a rapid profit if the price of two closely linked securities momentarily differs. Furthermore, the Securities and Exchange Commission (2020) states that some HFT firms employ structural strategies in an effort to take advantage of structural weaknesses in the market or in certain market players. This can be implemented by HFTs with the fastest computer programs with the lowest-latency market data and processing tools. Thanks to their technological advantage, such HFT firms may generate profit by trading with other market participants and traders who collect and process data more sluggishly and, in consequence, have not yet adjusted their prices to reflect the most recent events (Securities and Exchange Commission, 2020).

Latency arbitrage

Latency arbitrage can be referred to as an arbitrage strategy in which HFTs extensively employ speed to get in a dark pool[5] before the price updates, allowing them to trade for a better

[2] Auheuser-Busch InBev is one of the world's largest beer manufacturers. The group's activities are divided into two categories: beer production, and manufacture, bottling and sales of alcohol-free beverages (Euronext, 2022a).

[3] The BEL 20 is the most generally used indicator of the Belgian stock market. It is a free float market capitalization weighted index that reflects the performance of the 20 largest shares listed on Euronext Brussels (Euronext, 2022b).

[4] "*An American Depositary Receipt is a security that represents shares of non-U.S. companies that are held by a U.S. depositary bank outside the United States*" (Securities and Exchange Commission, 2012, p.1).

[5] Dark pools stand out from traditional trading venues for their limited or non-existent transparency, anonymity, and use of derivative pricing (Buti et al., 2011).

price. If the trade was made at updated prices, the investors in question would benefit from a more favourable price, which implies the advantage is gained at the expense of other market participants (Tichánek, 2020). Aquilina et al. (2021) bring forward that latency arbitrage has a detrimental influence on market liquidity and transactional costs. The strategy appears to be unethical and places investors at a disadvantage.

1.4.3 Directional strategies

In its recent Staff Report, the Securities and Exchange Commission (2020, p.40) states that "*directional strategies generally involve establishing a short-term long or short position in anticipation of a price move up or down*". To initiate such long or short positions, directional strategies usually require demanding liquidity. Some of those directional strategies aim to predict changes in price faster than other market participants. This necessitates sophisticated analytics and quick processing.

Jones (2013) further affirms that some HFT firms scan news releases electronically. By doing so, they perform textual analysis, and trade then on the inferred news. For example, such a program might look for the words like "raise" or "higher" or else "increased" near the phrase "earnings forecast". The algorithm next recognizes the business on which the news story is focused, and in this case briskly submit long position orders.

Some HFT firms make use of order flow indication to perform trades. Suppose a large buy order executes at the current ask price, an HFT approach might conclude the order submitter has significant positive information. The HFT could then replicate a competitor's behaviour by purchasing shares in turn (Jones, 2013).

Liquidity detection strategies

We should add that strategy based on order flow signals can be derived for large institutional traders. We therefore observe liquidity detection strategies, also referred to as order anticipation strategies (Hirschey, 2021). Liquidity detection strategies aim to detect big orders, or a series of smaller orders. For HFTs, the key is to anticipate these orders and trade ahead of them. HFT firms consequently seek information about an order that is going to be fulfilled on trading venues (Tichánek, 2020). Assume a large institutional trader progressively purchasing shares of Exxon Mobil[6] (ticker symbol: XOM). By scanning the market, HFT might be able to spot it by detecting a sequence of large buy orders within a short period of time. Hence, HFT firm could speedily get shares of XOM, moving up the price the institutional trader must pay to acquire more XOM shares. In these circumstances, an HFT firm could even turn a profit by selling its XOM shares to the institutional trader (Jones, 2013).

In an attempt to thwart these methods, institutional traders often use order splitting algorithms such as TWAP or VWAP we have reviewed earlier in Subsection 1.1.1. Traders strive to hide

[6]ExxonMobil is an integrated oil and gas business, searching for, mining, and refining oil all over the world. The corporation is among the top producers of both commodity and speciality chemicals and is the largest refiner in the world (Nasdaq, Inc., 2022a).

their overall trading intentions so that their orders appear to be from uninformed and sometimes ignorant regular investors. Financial markets then become the playground for hide-and-seek battle between programs and computers (Jones, 2013).

Now that we have set the context of HFT in this thesis, we will turn in the next chapter to the functioning and specifications of options. In particular, we will concentrate on Asian options. This will set the stage for Chapter 3 in which we will discuss different option pricing techniques, with a logical focus on the Asian option valuation.

Chapter 2

Options

In this second chapter, the general specifications and functioning of options are discussed in Section 2.1 in order to provide a theoretical overview of the central element of our research question, that is, optional derivatives. To clarify this, we investigate various option styles in Section 2.2, namely American options, European options, and Asian options which we will look at in more detail in subsequent chapters.

2.1 Definition of options

Options are derivative contracts that grant the buyer of the contract the right but with no obligation to buy or sell a specified quantity, at a premium, of an underlying asset at a specified price within a specified period or on a specified date. Reciprocally, the seller of an option will be obliged to honour its commitments if the buyer chooses to exercise. In doing so, an option allows the terms to be set today for a future transaction. Both parties to the contract will then have for one the right and for the other the obligation to complete the transaction. We can highlight two option mechanisms: calls and puts, which we will analyse later on (Fox, 2021). This right without obligation clearly differentiates options from other derivatives such as forward and futures, where the holder is compelled to buy or sell the underlying asset. Additionally, futures and forward contracts do not cost anything to enter into, whereas buying options involves an up-front payment (Hull, 2022). From the above description, we can underscore several characteristics inherent in an option:

- The **underlying asset** is an asset or variable on which the value of an option or derivative depends. Option contracts can be based on a multitude of underlying assets. The latter can be a financial asset such as a stock, a bond, a currency, or a physical asset such as a commodity. Hull (2022) points out different underlying assets for option contracts: stock options have stocks as underlying assets, ETP options use exchanged-traded products or funds, foreign currency options are based on currencies, index options are based on indexes and futures options uses derivative futures contracts as underlying asset. Naturally, the amount or the **quantity of the underlying asset** has to be unequivocally defined when concluding the contract.

- The **option premium** represents the cost for the option holder that has the right to buy or sell an underlying asset at a predetermined price. In order to obtain possession of the option, the buyer has to pay the premium to the seller. Therefore, it represents a financial compensation for the situation in which the seller of the contract has no rights but all obligations (Hull, 2022).
- The **option strike price** or exercise price is the price at which the underlying asset can be bought or sold. It is determined between both parties at the time of entering into the contract (Fox, 2021).
- The **option type** refers to the distinction between calls and puts. This has to be determined when concluding the contract in order to know the position of both parties (Hull, 2022). These positions are detailed below in Subsections 2.1.1 and 2.1.2.
- The **maturity date**, exercise date, or else expiration date, is also defined at the beginning of the option contract. It determines when the option expires. The date or period of exercise depends on the **option style** itself. Briefly, American options can be exercised at any time while European options can only be exercised on the predetermined exercise date (Hull, 2022). We will come back to the option styles later in Section 2.2.

All in all, the value of the option depends on various determining factors. The present value of the underlying asset, the time left to the option's expiration, the exercise price set by the contract, the risk-free interest rate as well as the volatility of the underlying asset price are key elements establishing the value of an option. Taking all these elements into account, there are multiple models that allow one to determine an option price. We will consider them in Chapter 3. Also, it is important to note that options may be traded both on exchanges and in the OTC market (Hull, 2022).

2.1.1 Call options

Hull (2022, p.9) defines a call option as a contract that "*gives the holder the right to buy the underlying asset by a certain date for a certain price*". A call option can be identified as a commitment to sell from the seller of the contract.

Because the buyer of the option is said to have a "long position" and the seller of the contract is said to have a "short position", two basis positions are possible while analysing a call option, referred to as long call or short call (Fox, 2021). There are consequently two sides to every option contract. Either the investor initiates a long position or he initiates a short position. Hence, profit or loss relating to the seller is logically opposite to the buyer (Hull, 2022). Positions regarding call options are explained below.

Long call

The investor that initiates a long position on a call expects the price of the underlying asset to rise. If he is wrong, he will realize a loss limited to the premium previously paid. On the other hand, if the investor's expectation is right, he will potentially get an unlimited gain when the price of the underlying asset exceeds the strike price. It should be noted that, from such a profit, it will still be necessary to subtract the premium which is paid in any cases. It is therefore possible to realize a net loss even if the underlying asset price exceeds the option strike price. This may happen when the price of the underlying asset does not go above the strike price sufficiently to cover the costs of the premium (Hull, 2022).

Consider the following example represented in Figure 2.1 which illustrates different positions for call and put options. The top-left graph shows a situation where an investor has a long position on a call option, or a long call. This figure depicts the option price to be $5 with a corresponding strike price of $100. Given that one contract is usually an agreement to buy or sell 100 shares, if the investor wants to purchase an option contract, the initial investment will be $500. Suppose the current underlying asset price is $95. As long as the latter is less than $100, the investor will obviously decide not to exercise his options. The initial investment of $500 will thereby be lost in its entirety. Alternatively, suppose the underlying asset price increases up to $110. By exercising the option, the investor is then able to buy 100 shares of the underlying asset for $100 per share. Ignoring transaction costs, selling those shares immediately will generate a profit of $10 per share for 100 shares, representing a total benefit of $1,000. When taking the initial cost of the option into consideration, the investor's net profit amounts to $500.

However, as mentioned above, an investor may occasionally exercise an option and lose money overall. Assume the underlying asset price increases from $95 to $102. The investor would tend to exercise, generating a profit of $2 per share, or $200 in total. Including the initial investment, the investor would nonetheless realize a loss of $3 per share, or $300 all in all. In these conditions, we might argue not to exercise the options. However, not exercising would lead to a loss of $500 altogether, represented as the initial investment, which is worse than a $300 loss (Hull, 2022).

In more general terms and without including the initial cost, the payoff from a long position in a call option is $max(S_T - K, 0)$ with S_T the price of asset at maturity T and K the strike price. This demonstrates the fact that the option will be exercised if $S_T > K$ and will not be exercised if $S_T \leq K$ (Hull, 2022). The payoff related to a long call position is represented in Figure 2.2 below.

Short call

On this side, a call seller anticipates the price of the underlying asset to increase little or not at all. If he is wrong, he will get an unlimited loss. Conversely, he will make a limited gain if he is right (Fox, 2021). The bottom-left chart of Figure 2.1 shows how gains and losses are represented on the seller's side of a call option.

Consider the example shown in Figure 2.1. In the same way as for the previous example regarding a long call position, the option price is $5 with a strike price of $100. For 100 shares of the underlying asset, the seller of that call will make a profit limited to $500 as long as the underlying price is less than $100. When the underlying asset price is between $100 and $105, the call seller still realizes a small gain, or no profit if the price is $105 exactly. However, as soon as the price soars over $105, the call seller will start losing money. This loss is unlimited and increases when the underlying asset goes up in price.

Generally speaking and excluding the initial cost, the payoff from a short position in a call option is $-max(S_T - K, 0) = min(K - S_T, 0)$ with S_T the price of asset at maturity T and K the strike price (Hull, 2022). The payoff related to a short call position is represented in Figure 2.2.

2.1.2 Put options

Hull (2022, p.9) defines a put option as a contract that "*gives the holder the right to sell the underlying asset by a certain date for a certain price*". A put option can be identified as a commitment to buy from the seller of the contract.

As with call options, long and short puts exist. An investor can therefore initiate either a long position or a short position (Hull, 2022). Those positions about put options are exposed below.

Long put

While a call buyer is anticipating that the price of the underlying asset will increase, the purchaser of a put is expecting it to fall. If an investor that initiated a long position on a put option is right, he will get a limited profit when the price of the underlying asset drops below the corresponding strike price. Otherwise, he will make a limited loss to the premium he paid (Hull, 2022).

Consider again the following example represented below which assumes a put option price of \$5 with a \$100 exercise price. The top-right chart in Figure 2.1 therefore illustrates a situation where an investor has a long position on a put option. Suppose this investor wants to sell 100 shares of the underlying asset. His initial investment consequently comes to \$500. The price of the corresponding underlying asset is \$97 on the day the options were purchased. In this instance, this investor could already be exercising his options. However, he will realize a small loss of \$2 per share when including the initial cost. When the underlying asset price is less than \$95, a profit can be made for the option holder. Suppose the price moved down to \$80. The investor can buy 100 shares for \$80 each and, under the terms of the put option, sell the same shares for \$100 to make a profit of \$20 per share, or \$2,000 overall. When including the \$500 premium, the generated profit amounts to \$1,500. In contrast, he will lose the \$500 premium when the underlying asset price climbs above \$100. Hence, the put option expires worthless (Hull, 2022).

More generally and still without including the initial cost, the payoff from a long position in a put option is $max(K - S_T, 0)$ with S_T the price of asset at maturity T and K the strike price. This demonstrates the fact that the option will be exercised if $S_T < K$ and will not be exercised if $S_T \geq K$ (Hull, 2022). The payoff related to a long position on a put option is represented in Figure 2.2.

Short put

A put seller predicts that the price of the underlying asset will rise or fall only slightly. Unlike a short position on call options, if a put seller is wrong, he incurs a limited loss. On the other hand, he will make a limited gain if the underlying asset price rises (Fox, 2021).

Consider Figure 2.1 below which illustrates the profit and loss for a short put holder in the bottom-right chart. Because the option price is \$5 with an exercise price of \$100, a put seller will lose money as soon as the underlying asset price drops under \$95. This loss increases when the price of the underlying asset moves down but is limited when the underlying reaches a price of \$0. For 100 shares of the underlying asset, a limited profit of \$500 is made when the underlying asset price increases to \$100 and above. The put seller realizes a smaller profit when the underlying asset price is less than \$100 but more than \$95.

Using a broader perspective and excluding the initial expense, the payoff from a short position in a put option is $-max(K - S_T, 0) = min(S_T - K, 0)$ with S_T the price of asset at maturity T and K the strike price (Hull, 2022). The payoff related to a short position on a put option is represented in Figure 2.2.

Figure 2.1: Basic option strategies.

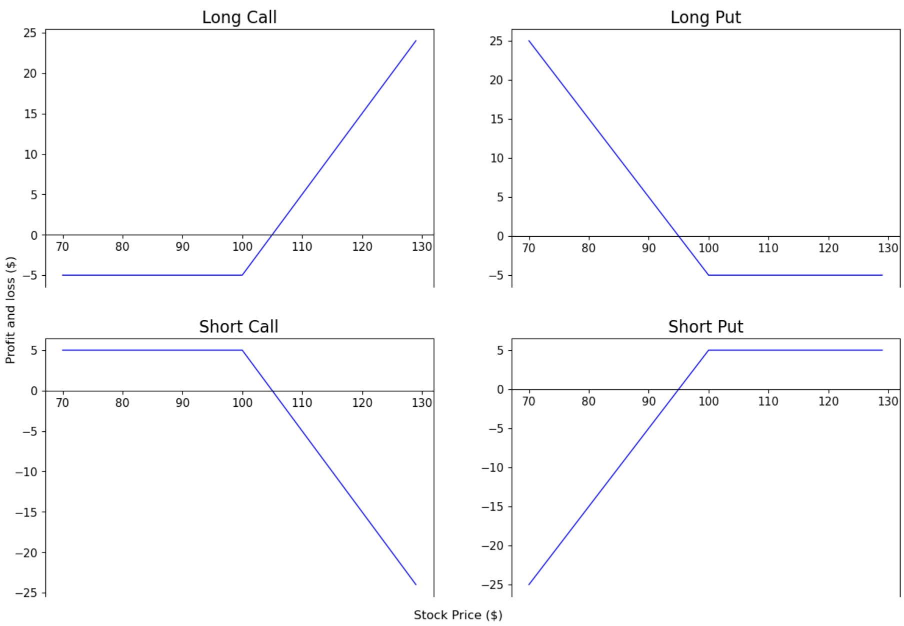
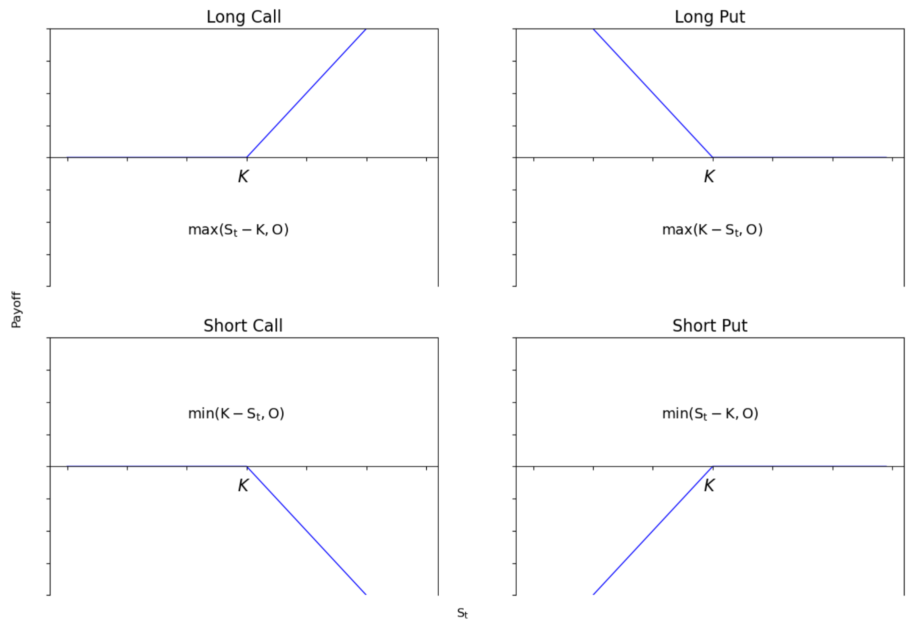


Figure 2.2: Payoffs from basic option strategies.



2.2 Various option styles

Although most of the options traded on financial markets are American-style options, there are other varieties of option (Fox, 2021). The best known options are European and American which we describe below in Subsections 2.2.1 and 2.2.2. In addition, so-called exotic options exist. Among the latter are Asian options which we describe further in Subsection 2.2.3. Note that the distinction between financial derivatives (American, European or Asian) has nothing to do with geographical location. Also, it should be mentioned that European options are generally simpler to analyse than American options (Hull, 2022). In Chapter 3 and 4 of this master thesis, we focus on the more complex options presented in this document, namely Asian options.

2.2.1 European options

Hull (2022, p.813) defines a European option as one "*that can be exercised only at the end of its life*". Consequently, an investor that holds European options can exercise his options at the maturity date or exercise date only (Fox, 2021).

Suppose a trader wants to purchase a call option on Meta Platforms, Inc.[7] (ticker symbol: META) with a strike price of \$170 and an exercise date in December 2022. Table 2.1 gives the bid and ask quotes for some of the call options trading on META, on June 17, 2022. At that time, bid and ask were respectively \$162.84 and \$162.94 for the META stock price (Chicago Board Options Exchange, 2022). Because one contract is generally worth 100 shares, an investor would pay a total of \$2,110 to acquire 100 shares of the underlying asset with a strike price of

[7]With more than 3.6 billion people who are active each month, Meta Plateforms, Inc. is the largest online social network in the world (Nasdaq, Inc., 2022b).

Table 2.1: META call option prices.

Strike price	*September 16, 2022*		*December 16, 2022*		*March 17, 2023*	
($)	*Bid*	*Ask*	*Bid*	*Ask*	*Bid*	*Ask*
145	27.30	28.05	33.25	34.00	37.55	38.55
150	24.25	24.95	30.35	31.10	34.75	35.75
155	21.50	21.90	27.80	28.25	32.25	32.95
160	18.90	19.20	25.35	25.70	29.90	30.30
165	16.55	16.80	23.00	23.35	27.45	27.95
170	14.30	14.55	20.80	21.10	25.25	25.70
175	12.25	12.50	18.75	19.00	23.20	23.60
180	10.45	10.70	16.70	17.05	21.25	21.65

Notes: Call option prices have been downloaded from the Chicago Board Options Exchange on June 17, 2022.

$170, or $21.10 per share. Assume the option is European-style, an investor will only be able to exercise his options in December. At the maturity date, he will choose to exercise if the META stock price has increased above $170. The seller of the option will have no choice but to sell META stocks at that price. Not surprisingly, the investor will not exercise if the META stock price is less than $170. In that case, he will prefer to buy META stocks directly on the equity market at a lower price (Hull, 2022).

2.2.2 American options

As opposed to European options, American options can be exercised at any time until the contract expires. Hull (2022, p.805) defines an American option as one "*that can be exercised at any time during its life*". Hence, the holder of several American options has the ability to exercise his options throughout the period until the maturity date. This is referred to as the exercise period (Fox, 2021).

Table 2.2: META put option prices.

Strike price	*September 16, 2022*		*December 16, 2022*		*March 17, 2023*	
($)	*Bid*	*Ask*	*Bid*	*Ask*	*Bid*	*Ask*
145	10.90	11.20	15.70	15.95	18.70	19.00
150	12.80	13.05	17.75	18.05	20.80	21.15
155	14.90	15.15	19.95	20.30	23.05	23.45
160	17.15	17.45	22.35	22.75	25.50	25.90
165	19.70	20.00	25.00	25.30	28.05	28.50
170	22.45	22.75	27.70	28.05	30.80	31.20
175	25.40	25.80	30.60	30.90	33.65	34.15
180	28.40	29.30	33.55	34.05	36.65	37.20

Notes: Put option prices have been downloaded from the Chicago Board Options Exchange on June 17, 2022.

Suppose now an investor wants to buy a put option on META with a strike price of \$150 and a maturity date in September 2022. Table 2.2 illustrates the bid and ask quotes for some of the put options trading on META, on June 17, 2022. At that time, bid and ask were respectively \$162.84 and \$162.94 for the META stock price (Chicago Board Options Exchange, 2022). As we know an option contract generally represents 100 shares, a trader will pay a sum of \$1,305, or \$13.05 per share. Since the options are American-style, the investor will be able to exercise at any moment up to September 16. Therefore, if the META stock price drops below \$150 before the maturity date, the investor will logically exercise his options and generate a gain. Otherwise, he will lose his initial investment of \$1,305, which will profit the seller of the options (Hull, 2022).

2.2.3 Asian options

As mentioned earlier, Asian options are exotic options. Hull (2022, p.805) defines Asian options as options "*with a payoff dependent on the average price of the underlying asset during a specified period*". The payoff then depends on the arithmetic, or geometric, average of the price of the underlying asset during the life of the option. Because Asian options are path-dependent, pricing them is much more complicated. Moreover, the "*arithmetic average is not lognormally distributed when the underlying follows a standard lognormal process. Hence, it is difficult to analytically derive the probability distribution*" (Tsao et al., 2003, p.488).

Also known as average options, Asian options are derivatives of stock options and are one of the most actively traded options in today's financial market.

The banking sector in Europe and America underwent a transformation in the 1980s. As a result of this, the Bank of England approved a statute in 1984 regulating foreign exchange option trading in the banking industry in the United Kingdom. Fixed income derivatives and related arbitrage operations were the focus of Mark Standish, a trader at Bankers Trust in London. At the same time, David Spaughton was working at a trust bank as a system analyst. Later, both Standish and Spaughton successfully collaborated in Tokyo, Japan, to establish an option formula for pricing crude oil average price options that could be utilized widely in the financial industry. This exotic option type is referred to as an Asian option and it represented an innovation compared to European options discussed earlier in Subsection 2.2.1. A characteristic which Asian options have in common with European options is that investors may only exercise the option contract on the expiration date. They are known as European-style Asian options. However, the distinction lies in the fact that European options' income is determined by the price of the underlying asset on the expiration date while Asian options' income is determined by the historical average price of the underlying asset (Han and Hong, 2022).

Asian options are popular in the option market because they employ an average value and have lower volatility, making them less expensive than traditional options (Tsao et al., 2003). Also, average options help stop financial market manipulations by averaging prices (Henderson, 2010a). Finally, they are often used in interest rate and foreign currency markets because they provide a more stable expiry price or strike price over contract periods than their counterparts (Hsu et al., 2020).

Note that Asian options can be divided into two main distinct types: Fixed Strike Asian Options and Floating Strike Asian Options. These are detailed below. In order to exactly define the average value used in an Asian option contract, it is necessary to carefully identify several criteria. As a result, figuring out how to integrate data points into the average value is beneficial. This relates to whether an arithmetic, geometric, or other more sophisticated average should be used. Furthermore, it is necessary to choose which data points will be used, as well as how many data points will be used for computing the average. We might utilize all quotations or just a portion of them (Han and Hong, 2022).

Arithmetic and geometric average Asian options

Arithmetic and geometric averages are the two most basic and often used forms of averages for computing the value of an Asian option (Henderson, 2010a). The arithmetic mean of prices is calculated by adding all prices with equal weights, dividing by the total number of prices, and indexing the result. Another frequent approach is index weighted average, which implies that before computing the average price, the new price is given a larger weight than the previous price in the form of exponential decrease rather than equal weight (Han and Hong, 2022). Nevertheless, Tsao et al. (2003) argue that Asian options are constructed almost exclusively by computing the arithmetic average.

Discrete and continuous sampling

Asian options can be broken down into several categories depending on how many data points were used to generate the average and whether the whole transaction price was used or merely a selection. The integral of the asset price can be used in the computation of the average value when the prices are closely grouped in a limited period of time (Henderson, 2010b). The outcome is thus a sampled average that is updated on a regular basis. This is known as continuous sampling. Alternatively, one can employ a few solid data points. For instance, we could use a smaller data set such as the closing price. This is referred to as discrete sampling (Han and Hong, 2022).

Standard and forward-starting Asian options

Discrete Asian options can in turn be separated into two types. This distinction is based on the beginning price and average price of the underlying asset in discrete circumstances. Standard Asian options represent a situation where the original price is included in the average price. On the other side, a forward-starting Asian option does not include the initial price in the average price (Han and Hong, 2022).

Fixed and floating strike Asian options

As mentioned before, Asian options can be separated into two main sorts based on their return upon maturity. Fixed Strike Asian Options are those in which the payoff is determined by the difference between the underlying asset's historical average price and a fixed strike price K. A Floating Strike Asian Option is one in which the return is determined by the difference between the underlying asset's historical average price and the price on the maturity date S_T (Han and Hong, 2022).

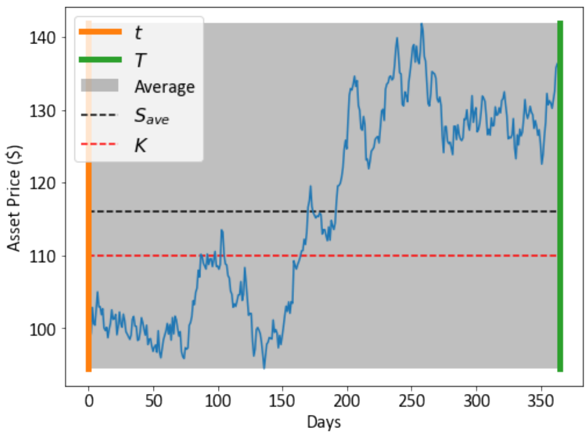
Fixed strike Asian options Under Asian options, the payoff from a Fixed Strike Asian Call Option is $max(0, S_{ave} - K)$ and that from a Fixed Strike Asian Put Option is $max(0, K - S_{ave})$ with S_{ave} being the historical average price of the underlying asset (Hull, 2022).

Sometimes named average price options, Fixed Strike Asian Options are typically less expensive than more traditional options and may be better appropriate than regular options for satisfying some of the needs of corporate treasurers. As a matter of example, consider the case of a U.S. corporate treasurer who expects a cash flow of one hundred million Australian dollars from the company's Australian division uniformly spread over the following year. He is likely to be interested in an option strategy that ensures the average exchange rate realized during the year is higher than a certain threshold. This can be done in a more efficient way with average price put options rather than regular put options (Hull, 2022).

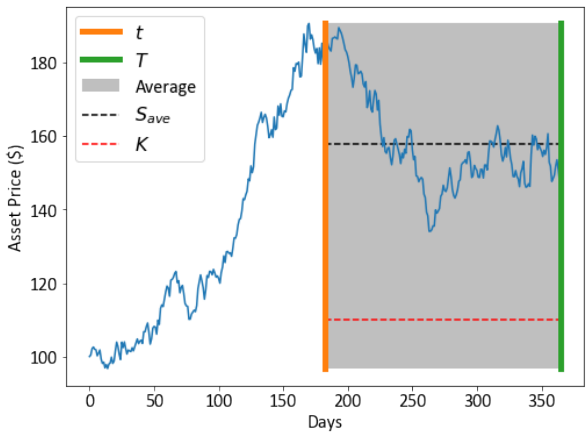
Figure 2.3 below presents two graphs that illustrate the period over which the average is calculated for a Fixed Strike Asian Option. Figure 2.3a shows a standard Asian option, while Figure 2.3b depicts a forward-starting Asian option. In both cases, we fixed the strike price K arbitrarily at \$110. Also, we priced the underlying asset at time $t = 0$ at \$100. We assume a year contains 365 days and the option expires in one year exactly. For the case of the forward-starting Asian option, we presume the average to be computed after six months, or from time $t = T/2$.

Figure 2.3: Fixed Strike Standard and Forward-Starting Asian Option.

(a) Fixed Strike Standard Asian Option



(b) Fixed Strike Forward-Starting Asian Option



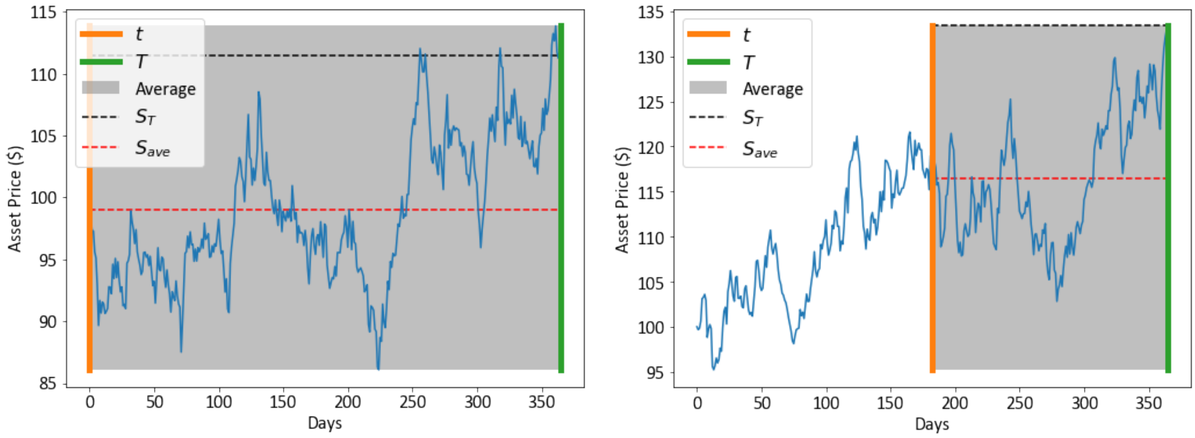
Floating strike Asian options Another sort of Asian option is average strike options, or Floating Strike Asian Options. As mentioned above, Floating Strike Asian Options are defined as options where "*the return depends on the difference between historical average price of the underlying asset and the price on the maturity date*" (Han and Hong, 2022, p.3). Consequently, the payoff from an average strike call is $max(0, S_T - S_{ave})$, while that of a Floating Strike Asian Put Option is $max(0, S_{ave} - S_T)$ where S_T is the price of the underlying asset at the maturity date T, indicated as final price or closing price. Average strike options can ensure that the average price paid for an asset in frequent trading over a period of time is not higher than the closing price. It can also guarantee that the average price received for an asset in frequent trading over a period of time does not fall below the final price (Hull, 2022).

Figure 2.4 displays two more graphs that represent the time frame over which the average is computed for an Asian option with a floating strike. On the left side, Figure 2.4a illustrates a standard average option. Figure 2.4b pictures that the average is computed after time $t = T/2$ in the case of a forward-starting Asian option. We still make the assumption that the option has a maturity T of 365 days and an initial price $S_0 = \$100$.

Figure 2.4: Floating Strike Standard and Forward-Starting Asian Option.

(a) Floating Strike Standard Asian Option

(b) Floating Strike Forward-Starting Asian Option



After setting the context in the previous chapter, we discussed here the central theoretical aspects of our research question, in particular Asian options. This provides us with all the necessary elements to examine various option pricing methods in the next chapter.

Chapter 3

Option Pricing Models

Undoubtedly, financial derivatives have become increasingly popular in recent years. One is not just using them as hedging instruments but also as speculative products. As a result, it is critical to use financial models that allow for accurate measurement of these financial derivatives' actual pricing. This craze has been supported by the progressive development of mathematical models. For instance, the original and renowned Black-Scholes model resulted in the development of option contract trading owing to its simplicity and comprehensiveness (Janková, 2018).

Having set the context for the research question and outlined the theory behind the options, we now apply some of the prior concepts to diverse option pricing approaches. Here we deal with the third central point of our research question, namely the valuation of call and put options. To maintain consistency with our research question, we will focus on the valuation of the Asian options. For this purpose, we go through three different option pricing models. In Section 3.1, we first briefly review the Binomial option pricing model through binomial trees. We then explore the Black-Scholes model in Section 3.2. The challenges with determining volatility in the original Black-Scholes model, as well as the continuous volatility assumption, are discussed in Subsection 3.2.2. We then investigate another but less accurate option pricing model, known as the Monte Carlo simulation, in Section 3.3. Finally, in Section 3.4, we apply these multiple option pricing techniques to the more concrete case of META, which we briefly looked at in Chapter 2.

All Excel files and Jupyter Notebooks mentioned in this and subsequent chapters are presented in a Google Drive folder[8]. Code and computations regarding various option pricing methods can be accessed through the following link: https://drive.google.com/drive/folders/1hQ0w0XIfaQo7HCl-kTZ_G_vD1A7bosSr?usp=sharing.

3.1 Binomial option pricing model

The binomial tree technique for pricing options, also known as CRR model after its inventors, was presented by John C. Cox, Stephen A. Ross and Mark Rubinstein in the late 1970s (Davis,

[8] We use the Python computer language for coding. Monte Carlo simulation was performed using a personal computer with the following specifications: M3-6Y30-900MHz with 4 GB RAM.

2010a). The model is a simple, straightforward, and adaptable pricing model for a variety of derivatives, which made it useful and popular (Coelho and Reddy, 2018).

The hypotheses proposed by Cox et al. (1979) are quite simple. Indeed, it is supposed that the underlying asset price can only move in two directions, that is, up or down. Moreover, this operation is repeated at each node, or at each time t. Cox et al. (1979) thus assume the underlying asset price to move up or down in a fixed proportion at predetermined time intervals. In order to determine the price of the underlying asset at the next node, the stock price is multiplied by the up factor (u) and the down factor (d). This calculation is constant throughout the tree. Following Zhang et al. (2017), we have:

$$u = exp(\sigma\sqrt{T/N}) \quad (3.1)$$

$$d = \frac{1}{u} = exp(-\sigma\sqrt{T/N}) \quad (3.2)$$

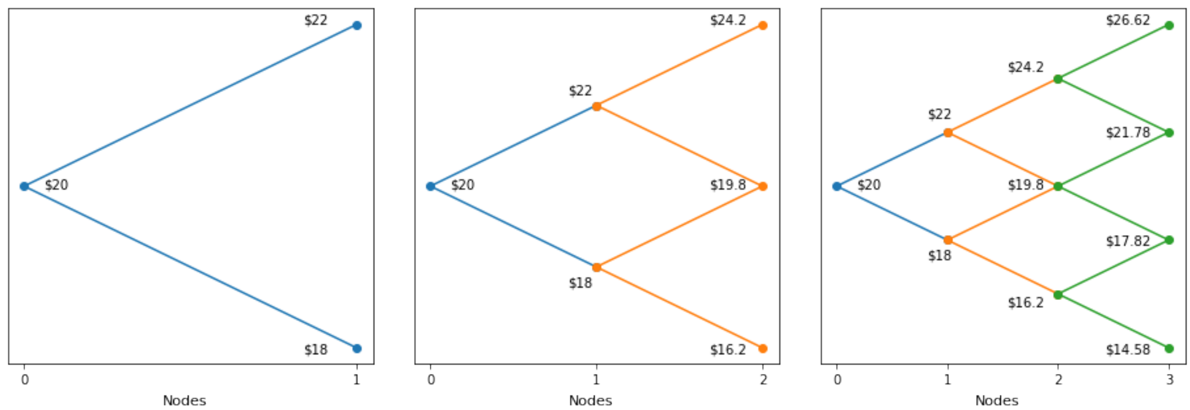
where N is the number of time steps modelled in the binomial tree, T is the option's expiration time, and σ is the stock price's annualised volatility.

We may consequently implement a N-step binomial tree and sample from the 2^N paths that are viable. This is represented by a diagram that depicts the many paths the stock price might follow over the life of an option (Davis, 2010a). Hence, the stock price follows a random walk, according to the underlying made assumptions. In each time step, it has a certain probability p of moving up by a certain percentage amount u and a certain probability $1 - p$ of moving down by a certain percentage amount d, according to Cox et al. (1979).

For instance, suppose a simple scenario proposed by Hull (2022) in which a stock is now priced at \$20. It is known that, in a one-step binomial tree, this stock price at the end of a 3-month period will be either \$22 or \$18. Assume a European call option on that specific underlying asset with a strike price of \$21. At the end of the three months, the option will have one of two values: either \$22 or \$18. If the stock price reaches \$22, using the call payoff formula we have studied in Chapter 2, the option's payoff will equal \$1. Otherwise, if the stock price decreases to \$18, the payoff will be zero. Figure 3.1 presents three binomial trees with distinct situations. Figure 3.1a illustrates the above-explained situation.

Figure 3.1: Binomial trees with one, two and three nodes.

(a) One-step binomial tree (b) Two-step binomial tree (c) Three-step binomial tree



The above analysis can then be extended to a two-step binomial tree. Briefly, consider again the stock price starts at \$20. In each of two-time steps, it may arbitrarily go up by 10% or down by 10%. Because there is a binomial stock price movement in each time step, when considering a two-step binomial tree, we have 2^2, i.e., 4 possible stock price paths that are implicitly considered. Figure 3.1b exemplifies such a situation. In the same way, Figure 3.1c shows a situation in which we expose a binomial tree with three nodes.

In any situations, the purpose of the analysis is to compute the option price at the initial node of the tree, $N = 0$ (Hull, 2022). As explained in the preceding chapter, the payoff from a European call option is determined at the binomial tree's final step by $max(S_T - K, 0)$. Option price at the penultimate node can then be calculated applying the following equation:

$$f = e^{-rdt}[pf_u + (1-p)f_d] \tag{3.3}$$

with

$$p = \frac{e^{-rdt} - d}{u-d} \tag{3.4}$$

where f is the option price, r is the annual risk-free rate, dt is the incremental time step, u is the up factor, d is the down factor and p is the risk-neutral probability (Fox, 2021).

This formula is again applied to each node, up to the initial node, $N = 0$. Similar formulas can be used in order to determine the value of the European put option where the payoff is determined by $max(K - S_T, 0)$. In the limit, as the number of time step is increased, this model converges to the Black-Scholes model which is discussed hereafter in Section 3.2 (Josheski and Apostolov, 2020).

In the end, it turns out that a rather simple argument may be used to price the option using this numerical valuation. Indeed, the only required assumption is based on the fact that there are no arbitrage opportunities (Hull, 2022).

3.2 Black-Scholes model

As we mentioned above, derivative contracts now account for a significant share of the financial market. Investors primarily trade those contracts because they are a basic mechanism for hedging various risks. The proliferation of these instruments enhanced the efficiency of the global financial market. It was facilitated by the gradual advancement of mathematical models. Because it is straightforward and clear, the Black-Scholes option pricing model became largely popular and contributed to a boom in option trading. Even though the price of the option computed using this model roughly matches the actual price, there are several flaws that remain. We will come back to these limitations in Subsection 3.2.2. Researchers have been working on it to address those issues with the aim of providing a more appropriate framework for financial derivatives pricing (Janková, 2018).

The Black-Scholes model is a well-known pricing mechanism that was initially developed to value European options. It was first derived and published in the *Journal of Political Economy* in 1973. The concept was developed by Fisher Black, Myron Scholes, and subsequently Robert

Merton (Davis, 2010b). In parallel with the binomial option pricing model, their idea was based on the fact that an option can be completely replicated by buying and selling the basic instrument and a risk-free asset in a specified proportion, hence removing the risk (Black and Scholes, 1973).

For European call and put options, the Black-Scholes formulas can be written as follow:

$$FV(Call) = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{3.5}$$

$$FV(Put) = -S_0 N(-d_1) + Ke^{-rT} N(-d_2) \tag{3.6}$$

where S_0 is the centre, current or spot price at time 0, T is the time to maturity of the option, K is the strike price, r is the continuously compounded risk-free interest rate and $N(*)$ is the Normal distribution function (Fox, 2021).

Accordingly, d_1 and d_2 can be obtained applying following formulas:

$$d_1 = \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right) \tag{3.7}$$

$$d_2 = \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right) = d_1 - \sigma \sqrt{T} \tag{3.8}$$

where σ is the stock price volatility (Josheski and Apostolov, 2020).

The existence of a derived model should also be addressed. Actually, shortly after the Black-Scholes model was issued, the Merton model was released, sometimes referred to as the Black-Scholes-Merton model. The main distinction between the two models lies in the fact that the Merton model also accounts for dividends, which are not included in the original Black-Scholes model (Merton, 1976). Nonetheless, the later has not gained as much traction as the original model, specifically because it assumes a constant dividend payment. This constraining assumption makes the model unrealistic (Janková, 2018). We obtain the price of a European call c and a European put p on a stock that pays a dividend yield by replacing S_0 by $S_0 e^{-qT}$ in former Equations 3.5 and 3.6, so that:

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2) \tag{3.9}$$

$$p = -S_0 e^{-qT} N(-d_1) + Ke^{-rT} N(-d_2) \tag{3.10}$$

where S_0 is the centre, current or spot price at time 0, T is the time to maturity of the option, K is the strike price, r is the continuously compounded risk-free interest rate, q is the dividend yield, and $N(*)$ is the Normal distribution function (Hull, 2022).

Following Merton (1976), d_1 and d_2 are hence determined by:

$$d_1 = \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right) \tag{3.11}$$

$$d_2 = \left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \right) = d_1 - \sigma \sqrt{T} \tag{3.12}$$

3.2.1 Black's formulas for Asian option pricing

If we assume S_{ave} to be lognormal, average price options can be valued applying similar methods and formulas to those used for standard options. In fact, this is a reasonable assumption when the standard model for the process followed by the asset price is made. A common solution for valuing fixed strike Asian options is to employ Black's formulas and to fit a lognormal distribution to the first two moments of S_{ave} (Hull, 2022).

Replacing S_0 by F_0 and q by r in Equations 3.9 and 3.10, we obtain:

$$c = e^{-rT}\left[F_0 N(d_1) - KN(d_2)\right] \tag{3.13}$$

$$p = e^{-rT}\left[-F_0 N(-d_1) + KN(-d_2)\right] \tag{3.14}$$

with

$$d_1 = \left(\frac{\ln\left(\frac{F_0}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) \tag{3.15}$$

$$d_2 = \left(\frac{\ln\left(\frac{F_0}{K}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) = d_1 - \sigma\sqrt{T} \tag{3.16}$$

where F_0 is the futures price at time 0, T is the time to maturity of the option, K is the strike price, r is the continuously compounded risk-free interest rate, σ is the volatility of the futures price, and $N(*)$ is the Normal distribution function (Hull, 2022).

Imagine M_1 and M_2 to be the first two moments of S_{ave}. Following Hull (2022), we have:

$$F_0 = M_1 \qquad \text{and} \tag{3.17}$$

$$\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right) \tag{3.18}$$

When the average is calculated continuously, and r, q, and σ are constant, we obtain:

$$M_1 = \frac{e^{(r-q)T}-1}{(r-q)T}S_0 \tag{3.19}$$

$$M_2 = \frac{2e^{\left[2(r-q)+\sigma^2\right]T}S_0^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0^2}{(r-q)T^2}\left(\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right) \tag{3.20}$$

where S_0 is the price of the underlying asset at time 0, T is the time to maturity of the option, r is the continuously compounded risk-free interest rate, q is the dividend yield, and σ is the volatility of the futures price (Hull, 2022).

Broadly speaking, the average from observations at times T_i $(1 \le i \le m)$ is determined with:

$$M_1 = \frac{1}{m}\sum_{i=1}^{m} F_i \tag{3.21}$$

$$M_2 = \frac{1}{m^2}\left(\sum_{i=1}^{m} F_i^2 e^{\sigma_i^2 T_i} + 2\sum_{j=1}^{m}\sum_{i=1}^{j-1} F_i F_j e^{\sigma_i^2 T_i}\right) \tag{3.22}$$

where F_i is the forward price and σ_i is the implied volatility for maturity T_i (Hull, 2022).

3.2.2 Limitations of the original Black-Scholes model

First limitation of the original model stems from the fact that the absence of arbitrage is a necessary condition for it to solve the differential equation. Though, this is commonly violated and often leads to anomalies. Another prerequisite for the model's derivation stands in the perfect derivative replication by the underlying asset and a certain percentage of a risk-free instrument. This can however not be completed without transaction costs. Furthermore, although the Merton model solved the issue sometime later, the original Black-Scholes model does not account for the payment of dividends for the underlying share. Nevertheless, one is aware that dividends are paid by most stock corporations (Janková, 2018).

The most important and often debated parameter in the Black-Scholes model is without any doubt the volatility of the underlying asset returns. Indeed, the model presumes constant volatility, which also implies constant underlying asset returns (Ye, 2022). As a result, correctly determining volatility is essential for proper option contract pricing. This can be estimated using historical prices and returns of the underlying asset or based on option contract prices, referred to as implied volatility. However, volatility in the model is assumed to remain constant in both cases since it is stated by a single value characteristic for a given underlying asset (Janková, 2018).

Such constraints and assumptions explain in particular that the option values computed with the Black-Scholes model do not thoroughly match the actual option price. We will see in Section 3.4 that non negligible price differences arise when applying the Black-Scholes model to a practical case. To compensate for some of the Black-Scholes model's constraints, changes to market assumptions, particularly regarding continuous volatility and known risk-free interest rate, should be made (Krznaric, 2016).

3.3 Monte Carlo simulation

The Monte Carlo simulation is a mathematical financial approach for calculating the value of an option with a variety of uncertainties and random characteristics. Because this features are also inherent in gambling, this strategy is named after the city of Monte Carlo, Monaco, which is renowned for its casinos (Lu, 2011). It employs risk-neutral valuation results when valuing an option. We sample paths in order to compute the expected payoff in a risk-neutral environment and discount it at the risk-free rate. The simulation generates a vast number of probable outcomes as well as their likelihoods. The number of trials used in a Monte Carlo simulation determines the accuracy of the outcome. The more trials there are, the more accurate the simulation (Hull, 2022).

The primary benefit of using a Monte Carlo simulation is that it may be used in situations where the payoff depends on the path followed by the underlying asset price, which is what we are interested in, and in situations where it only depends on the underlying asset final value (Hull, 2022).

3.3.1 Monte Carlo simulation with constant volatility settings

As a starting point, a standard geometric Brownian motion is used to express the price of the underlying asset S_t. Under constant volatility conditions, a stochastic process S_t is therefore said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation (SDE):

$$dS_t/S_t = \mu dt + \sigma dW_t \tag{3.23}$$

where S_t is the underlying asset price at time t, μ is the expected rate of return per unit of time from the stock, σ is the stock price volatility. Additionally, W_t is a standard Wiener process[9]. In this section, we assume μ and σ are constants (Hull, 2022).

Considering a Brownian motion trajectory that satisfy this SDE, the right-hand side term μdt is the expected value of the return while σdW_t controls the stochastic component of the return, that is, the random noise of the trajectory. Equation 3.23 is the most widely used model of stock price behavior. The model represents the stock price process in the real world. For a non-dividend-paying stock in a risk-neutral world, μ equals the risk-free rate (Hull, 2022).

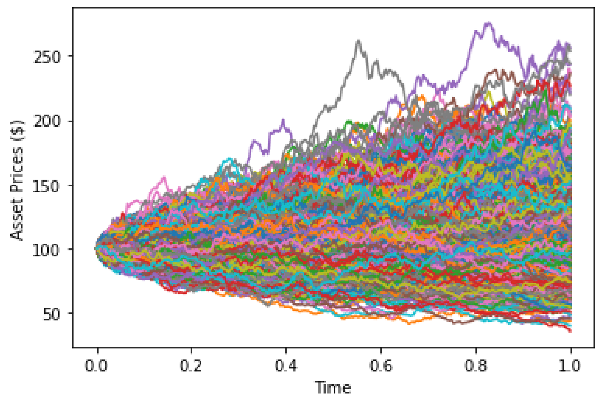
The solution to the above SDE can be obtained by taking the integration of both sides. We take the exponential afterwards, and incorporate the initial condition S_0. According to Agustini et al. (2018) ,we thus obtain the analytical solution for the standard geometric Brownian motion as follows:

$$S_t = S_0 e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]} \tag{3.24}$$

Among constant volatility models, we find the CRR model mentioned in Section 3.1, or the well-known Black-Scholes model addressed in Section 3.2.

As an illustration, Figure 3.2 hereafter shows an example of a standard geometric Brownian motion for an underlying asset with an initial price of \$100. This was performed arbitrarily with 1,000 simulations over a one-year time frame.

Figure 3.2: Geometric Brownian motion with constant volatility.



3.3.2 Monte Carlo simulation with stochastic volatility settings

Since it is unrealistic to assume that drift and volatility would remain constant over a long time period, we need to add a random process for the underlying asset volatility (Lidén, 2018). This

[9] A Wiener process, named after Norbert Wiener, is a continuous-time stochastic process. It is used to model Brownian motion (Lalley, 2016).

process also follows a geometric Brownian motion (Hull, 2022). Following the volatility model of Hull and White (1987), we therefore add to the previous hypothesis that the price of the underlying asset with stochastic volatility is calculated as follows:

$$dS_t/S_t = (r - d)dt + \sigma_t dZ_t \quad \text{and} \tag{3.25}$$

$$d\sigma_t^2/\sigma_t^2 = \mu dt + \xi dW_t \tag{3.26}$$

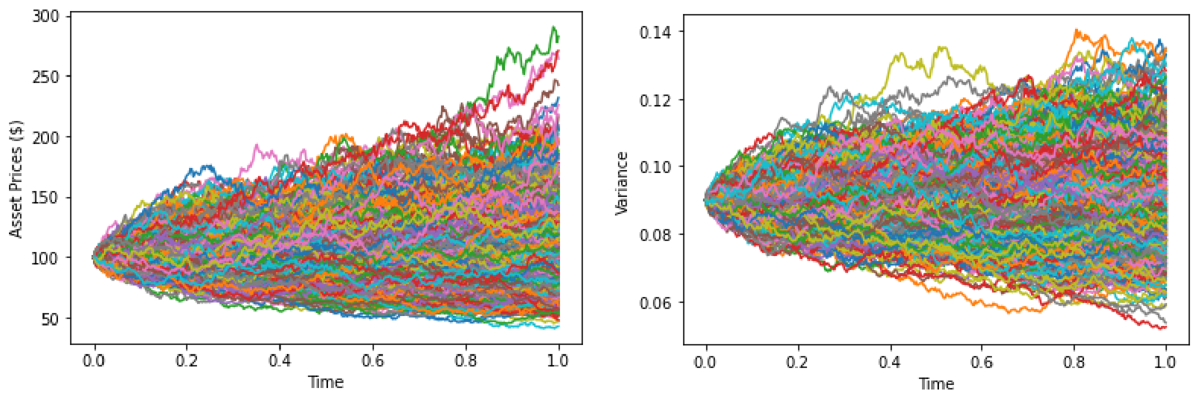
where S_t is the underlying asset price at time t, r is the annualized risk-free rate, d is the continuous dividend yield, σ_t denotes the instantaneous volatility of the underlying asset at time t, μ is the drift term, and ξ is the volatility term of the variance of the underlying asset. Both of the latter terms, μ and ξ, stay unchanged over time. Besides, Z_t and W_t are two independent Wiener processes (Hsu et al., 2020; Lin and Chang, 2020).

Below, Figure 3.3 illustrates two geometric Brownian motion for stochastic volatility settings. Using Equations 3.25 and 3.26, we simulated asset prices in Figure 3.3a. Figure 3.3b exemplifies a geometric Brownian motion for the variance process. In the same way as for the Monte Carlo simulation with Black-Scholes settings, we randomly set the underlying asset initial price such as $S_0 = \$100$. We also assume the option expires in one year, and we executed the Monte Carlo method with 1,000 simulations.

Figure 3.3: Geometric Brownian motion with stochastic volatility.

(a) Hull-White model asset prices

(b) Hull-White model variance process



3.4 Numerical applications

In that chapter's last section, we apply above-named option pricing models to the more specific and already studied case of META. In a comprehensive manner, we use an initial price $S_0 = \$162.95$ on June 17, 2022, with a strike price $K = \$170$ (Chicago Board Options Exchange, 2022). Also, the annual risk-free interest rate came to 2.24% on June 16, 2022 (U.S. Department of the Treasury, 2022). From Yahoo! Finance (2022), we downloaded META's historical quotes from December 31, 2020 to December 31, 2021. We then computed the daily volatility which equals to 1.86% and the annualized volatility which amounts to 29.52%. Note that META is a non-dividend-paying stock so that $d = 0\%$.

3.4.1 Binomial model application

Implementation in Excel Applying the simple two-step situation considered in Section 3.1 to META, we find the stock price with a value of \$188.83 or \$140.61 at the end of the first node ($N = 1$). It can then increase further to a value of \$218.83, return to the original price, or decrease again to a lower value of \$121.34.

Using Equations 3.1 to 3.4, we compute the European call option value on META to be \$11.22 while the European put option is worth \$16.39. This has also been computed for situations in which $N = 4$ and $N = 6$. Results are compared in the following table:

Table 3.1: META option prices, using small binomial trees.

	(1)	**(2)**	**(3)**
	BT(2)	**BT(4)**	**BT(6)**
Call	11.224044	11.520819	11.567144
Put	16.385825	16.682600	16.728925

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, and time to maturity = 0.498630. Option prices in columns (1), (2), and (3) are derived by applying a binomial tree with 2, 4, and 6 nodes respectively. See Excel file and Notebook *META option prices, using small binomial trees* for details.

Implementation in Python Implementing previously described binomial tree option pricing method in Python allows generating trees with a large number of nodes. Results for 10, 100 and 252 steps are presented in Table 3.2. With 252 time steps, there are 2^{252} possible stock price paths that are considered. Moreover, it can be noticed that the more steps in the binomial tree, the more accurate the value becomes.

Table 3.2: META option prices, using large binomial trees.

	(1)	**(2)**	**(3)**
	BT(10)	**BT(100)**	**BT(252)**
Call	11.553552	11.284054	11.326378
Put	16.715333	16.445835	16.488159

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, and time to maturity = 0.498630. Option prices in columns (1), (2), and (3) are derived by applying a binomial tree with 10, 100, and 252 nodes respectively. See Notebook *META option prices, using large binomial trees* for details.

3.4.2 Black-Scholes model application

European options

Applying previous Equations 3.7 and 3.8 for d_1 and d_2, we obtain $d_1 = -0.04534$ and $d_2 = -0.25383$. We then calculate the fair value of call and put options. Results are exhibited in

Table 3.3. Truly, buying a call on META expiring on December 16, 2022 with a strike price of \$170 would actually cost an investor \$21.10, as we have seen in Chapter 2 while the theoretical model is pricing the same call at \$11.31. In an analogous way, buying a put option on META with same parameters would cost \$28.05 whereas the Black-Scholes model prices the put option at \$16.48 instead.

Table 3.3: META option prices, using the Black-Scholes model.

	(1)
	Black-Scholes
Call	11.314635
Put	16.476416

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, time to maturity = 0.498630, $d_1 = -0.04534$, and $d_2 = -0.25383$. Column (1) displays call and put values obtained using the Black-Scholes model. See Excel file and Notebook *META option prices, using the Black-Scholes model* for details.

Asian options

Consider a Fixed Strike Asian Call Option on META. If the average is computed continuously, we have $M_1 = 163.86$ and $M_2 = 27,245.61$, from Equations 3.19 and 3.20. Using Equations 3.17 and 3.18, we have $F_0 = 163.86$ and $\sigma = 17.10\%$. This allows to compute d_1 and d_2 which are used to determine the value of the call option as \$5.2784 and the value of the put option as \$11.3468, using Equations 3.13 to 3.16. Results are presented in Table 3.4.

Table 3.4: META fixed strike Asian option prices.

	(1)
	Asian option
Call	5.278376
Put	11.346799

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, time to maturity = 0.498630, $d_1 = -0.24409$, and $d_2 = -0.36484$. Column (1) displays Asian call and put values obtained using Black's formulas. See Excel file and Notebook *META fixed strike Asian option prices* for details.

3.4.3 Monte Carlo simulation application

European options

Implementation in Excel Monte Carlo Estimates (MCEs) have been calculated under 1,000, 5,000 and 10,000 simulations which aims to improve exactness when we increase the number of simulations. Results for each MCEs are displayed in Table 3.5. Comparing the obtained Monte Carlo values with the value proposed by the Black-Scholes model in Subsection 3.4.2, it can be confirmed that as the number of simulations increases, the estimated value approaches that calculated with the Black-Scholes model. Therefore, within the case of the MCE(10,000), the resulting call and put option values approach the fair value computed with the original Black-Scholes model, with a relative error of 3.268×10^{-3} and 3.690×10^{-4} respectively. According to Lu (2011), the relative error is calculated as follows:

$$\text{Relative error} = \frac{\text{Measured} - \text{Benchmark}}{\text{Benchmark}} \tag{3.27}$$

However, MCEs are still relatively far from those proposed by the previous model. To overcome this inaccuracy, even more simulations are needed.

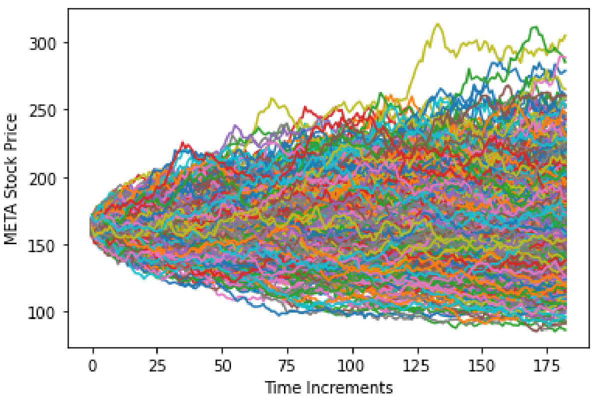
Table 3.5: META option prices, using a small Monte Carlo simulation.

	(1)	(2)	(3)	(4)
	Black-Scholes	**MCE(1,000)**	**MCE(5,000)**	**MCE(10,000)**
Call	11.314635	11.184538 (-1.1498%)	10.955642 (-3.1728%)	11.351610 (0.3268%)
Put	16.476416	15.936523 (-3.2768%)	16.623678 (0.8938%)	16.482494 (0.0369%)

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, time to maturity = 0.498630. Column (1) displays call and put values obtained using the Black-Scholes model. Option prices in column (2), (3), and (4) are derived by applying a Monte Carlo simulation method with 1,000, 5,000, and 10,000 paths respectively under constant volatility conditions. Relative errors are in parentheses. See Excel file *META option prices, using a small Monte Carlo simulation* for details.

Implementation in Python With an eye to improving accuracy, we programmed a Monte Carlo simulation in Python. This implementation is needed to increase the number of simulations as Excel is not capable of handling thousands of rows simultaneously in an efficient way.

Figure 3.4: META standard geometric Brownian motion for a European option.



Using a Jupyter Notebook allows one to generate millions of simulations concurrently. Results for 1,000, 10,000, 100,000 and 1,000,000 simulations are compared to Black-Scholes call and put option values in Table 3.6. It can be observed that the one million simulations Monte Carlo

Estimate, MCE(1,000,000), gets very close to the fair value of call and put option computed with the original Black-Scholes model, with low relative errors.

Table 3.6: META option prices, using a large Monte Carlo simulation.

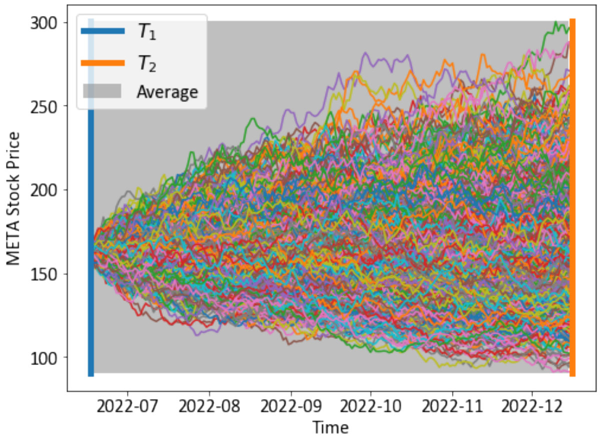
	(1)	(2)	(3)	(4)	(5)
	Black-Scholes	**MCE(1,000)**	**MCE(10,000)**	**MCE(100,000)**	**MCE(1,000,000)**
Call	11.314635	11.366217	11.271742	11.305942	11.282243
		(0.4559%)	(-0.3791%)	(-0.0768%)	(-0.2863%)
Put	16.476416	15.988384	16.237872	16.395567	16.477697
		(-2.9620%)	(-1.4478%)	(-0.4907%)	(0.0078%)

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, time to maturity = 0.498630. Column (1) displays call and put values obtained using the Black-Scholes model. Option prices in column (2), (3), (4), and (5) are derived by applying a Monte Carlo simulation method with 1,000, 10,000, 100,000, and 1,000,000 paths respectively under constant volatility conditions. Relative errors are in parentheses. See Notebook *META option prices, using a large Monte Carlo simulation* for details.

Asian options

Implementation in Python Suppose again a Fixed Strike Standard Asian Option where the underlying asset average price is computed arithmetically, between T_1 and T_2 on Figure 3.5.

Figure 3.5: META standard geometric Brownian motion for an Asian option.



Results for 1,000, 10,000, and 100,000 simulations are compared to previously computed Asian call and put option values in Subsection 3.4.2. MCEs are displayed hereafter:

Table 3.7: META fixed strike Asian option prices, using a Monte Carlo simulation.

	(1)	(2)	(3)	(4)
	Asian option	**MCE(1,000)**	**MCE(10,000)**	**MCE(100,000)**
Call	5.278376	5.158254	5.352011	5.350624
		(-2.2757%)	(1.3950%)	(1.3687%)
Put	11.346799	10.996704	11.595602	11.373588
		(-3.0854%)	(2.1927%)	(0.2361%)

Notes: The parameters are set as follows: $S_0 = 162.95$, $K = 170$, $r = 2.24\%$, $d = 0\%$, daily volatility = 1.86%, annualized volatility = 29.52%, time to maturity = 0.498630. Column (1) displays call and put values obtained using Black's formulas. Option prices in column (2), (3), and (4) are derived by applying a Monte Carlo simulation method with 1,000, 10,000, and 100,000 paths respectively under constant volatility conditions. Relative errors are in parentheses. See Notebook *META fixed strike Asian option prices, using a Monte Carlo simulation* for details.

Chapter 4

High Frequency Pricing of Asian Options

We have seen in Chapter 2 that Asian options, also known as average options, are path-dependent derivatives. In other words, the payoff from Asian options relies on the average value of the underlying asset within a specific set of dates in the life of the option. This feature greatly complicates the pricing of Asian options, compared to European and American options (Tsao et al., 2003). Besides, we know that Asian call and put options may be constructed using two basic forms of average, i.e., arithmetic or geometric average. In contrast to arithmetic average options, a closed-form solution exists for pricing and hedging a geometric average Asian option. Because the geometric average of lognormal random variables persists lognormal, it is consequently possible to obtain a closed-form solution for a geometric Asian option in a simple approach by extending the Black-Scholes model. Although Asian options with arithmetic average are the most popular and frequently used, there is no exact analytic solution for this type of exotic options. The main reason for this is that the arithmetic average of a set of lognormal random variables is not lognormally distributed (Lin and Chang, 2020).

Hence, it is challenging to develop a practical pricing formula for arithmetic Asian options. There are three main reasons for this. First, as noted above, Asian options are path-dependent. This implies that both the value of the underlying asset at time t and the history of the underlying asset are considered when computing the price of an Asian option. Second, it is difficult to calculate the probability distribution analytically when the underlying asset follows a standard lognormal process since the arithmetic average is not lognormally distributed. Third, stochastic volatility is extensively observed and is inherent in financial asset prices. For this reason, a dynamic stochastic volatility mechanism must be included in the pricing of Asian options for them to be useful in practice (Lin and Chang, 2020).

In this fourth chapter, we seek to replicate the study of Hsu et al. (2020). This paper extends the research of Hull and White (1987) to include a Taylor series expansion technique for deriving the approximate analytical solution of a Forward-Starting Asian Option (FSAO) in the context of stochastic volatility high frequency pricing. Hsu et al. (2020) therefore provided an analytical solution for a stochastic-volatility FSAO (SVFSAO). Additionally, we incorporate the study of Lin and Chang (2020) who observe standard Asian options under similar volatility conditions.

This thesis' fourth chapter is structured as follows. The problem being addressed is described in Section 4.1 along with the development of the approximate analytic solutions with Taylor expansion. Pricing performances of both models are then discussed in Section 4.2 in comparison with the benchmark of the Monte Carlo simulation solutions.

4.1 Materials and methods

Generally speaking, we consider hereafter a European-style Asian option written on an asset with maturity date T (Hsu et al., 2020; Lin and Chang, 2020). Even though we will focus on a call option, the results of the subsequent analyses are applicable without much effort to a European-style Asian put through the use of put-call parity (Hull and White, 1987).

According to Lin and Chang (2020), it was assumed that the standard Asian option maintains an arithmetic average strike price K over the period $[0, T]$. Consequently, the strike price of a standard average option is represented by the following equation:

$$K = \frac{1}{T} \int_0^T S_t \, dt \tag{4.1}$$

Because a FSAO can be regarded as a special type of Asian option in which the arithmetic average price is calculated from a contract's specific point in time, Hsu et al. (2020) assume, based on Tsao et al. (2003)'s former study, that the strike price K is computed over the period $[T - A, T]$. Thus, the exercise price of a FSAO is

$$K = \frac{1}{A} \int_{T-A}^T S_t \, dt \tag{4.2}$$

where $T - A$ is the time instant after time zero of the option's issuance.

For a FSAO, it is therefore fundamental to distinguish both time windows $[0, T - A]$ and $[T - A, T]$. Investors actually have no information about K prior to $T - A$. After time $T - A$, information about the strike price steadily builds up (Hsu et al., 2020). For the case in which time t is between time windows $T - A$ and T ($T - A < t \leq T$), investors have some information about the underlying asset price between $[S_{T-A}, S_t]$. According to Hsu et al. (2020), the exercise price K can thus be divided into two parts using the following formulas:

$$K = M_t + \frac{1}{A} \int_t^T S_t \, dt \tag{4.3}$$

where

$$M_t = \frac{1}{A} \int_{T-A}^t S_t \, dt \tag{4.4}$$

Additionally, Hull and White (1987) define the mean variance over some time interval $[0, T]$ by

$$\bar{V} = \frac{1}{T} \int_0^T \sigma_u^2 \, du \tag{4.5}$$

4.1.1 Standard Asian Options approximate analytic solution

The approximate analytic solution for pricing a standard Asian option with stochastic volatility has been developed by Lin and Chang (2020) using a Taylor series expansion technique so that:

$$\begin{aligned}
f(S_0,\sigma_0^2) \cong\ & S_0 e^{-rT}\left[m' N\left(\frac{m'}{\sqrt{v'}}\right)+e^{-\frac{m'^2}{2v'}}\sqrt{\frac{v'}{2\pi}}\right] \\
&+\frac{1}{2}\Bigg\{S_0 e^{-rT}\Bigg[M_2 N\left(\frac{m'}{\sqrt{v'}}\right)+\frac{1}{\sqrt{2\pi}}e^{-\frac{m'^2}{2v'}}M_1 A-\frac{1}{\sqrt{32\pi}}e^{-\frac{m'^2}{2v'}}v'^{-\frac{3}{2}}V_1^2 \\
&+\frac{1}{\sqrt{8\pi}}e^{-\frac{m'^2}{2v'}}v'^{-\frac{1}{2}}V_1 B+\frac{1}{\sqrt{8\pi}}e^{-\frac{m'^2}{2v'}}v'^{-\frac{1}{2}}V_2\Bigg]\Bigg\}\left(\frac{2\sigma_0^4(e^k-k-1)}{k^2}-\sigma_0^4\right) \\
&+\frac{1}{6}\Bigg\{S_0 e^{-rT}\Bigg[M_3 N\left(\frac{m'}{\sqrt{v'}}\right)+\frac{1}{\sqrt{2\pi}}e^{-\frac{m'^2}{2v'}}\alpha_1-\frac{1}{\sqrt{32\pi}}e^{-\frac{m'^2}{2v'}}\alpha_2 \\
&+\frac{1}{\sqrt{8\pi}}e^{-\frac{m'^2}{2v'}}(\alpha_3+\alpha_4)\Bigg]\Bigg\}\left(\sigma_0^6\left[\frac{e^{3k}-(9+18k)e^k+(8+24k+18k^2+6k^3)}{3k^3}\right]\right)
\end{aligned} \tag{4.6}$$

where $N(*)$ is the cumulative density function of standard normal distribution and $k=\xi^2 T$. Details regarding Equation 4.6 are available in Appendix B.

4.1.2 Forward-Starting Asian Options approximate analytic solution

With regards to forward-starting Asian options, a similar formula has been proposed by Hsu et al. (2020) who developed an approximate analytic solution using the formula below. In a comparable way, they integrated a Taylor series expansion into the Hull and White (1987) model.

$$\begin{aligned}
f(S_0,\sigma_0^2) \cong\ & S_0 e^{-r(T-t)}\left[m_2 N\left(\frac{m_2}{\sqrt{v_2}}\right)+e^{-\frac{m_2^2}{2v_2}}\sqrt{\frac{v_2}{2\pi}}\right] \\
&+\frac{1}{2}\Bigg\{S_0 e^{-r(T-t)}\Bigg[M_{22} N\left(\frac{m_2}{\sqrt{v_2}}\right)+\frac{1}{\sqrt{2\pi}}e^{-\frac{m_2^2}{2v_2}}M_{21}\bar{A}_2-\frac{1}{\sqrt{32\pi}}e^{-\frac{m_2^2}{2v_2}}v_2^{-\frac{3}{2}}V_{21}^2 \\
&+\frac{1}{\sqrt{8\pi}}e^{-\frac{m_2^2}{2v_2}}v_2^{-\frac{1}{2}}V_{21}\bar{B}_2+\frac{1}{\sqrt{8\pi}}e^{-\frac{m_2^2}{2v_2}}v_2^{-\frac{1}{2}}V_{22}\Bigg]\Bigg\}\left(\frac{2\sigma_0^4(e^k-k-1)}{k^2}-\sigma_0^4\right) \\
&+\frac{1}{6}\Bigg\{S_0 e^{-r(T-t)}\Bigg[M_{23} N\left(\frac{m_2}{\sqrt{v_2}}\right)+\frac{1}{\sqrt{2\pi}}e^{-\frac{m_2^2}{2v_2}}\alpha_{21}-\frac{1}{\sqrt{32\pi}}e^{-\frac{m_2^2}{2v_2}}\alpha_{22} \\
&+\frac{1}{\sqrt{8\pi}}e^{-\frac{m_2^2}{2v_2}}(\alpha_{23}+\alpha_{24})\Bigg]\Bigg\}\left(\sigma_0^6\left[\frac{e^{3k}-(9+18k)e^k+(8+24k+18k^2-6k^3)}{3k^3}\right]\right)
\end{aligned} \tag{4.7}$$

where $N(*)$ is again the cumulative density function of standard normal distribution and $k=\xi^2 T$. Details regarding Equation 4.7 are available in Appendix C.

4.2 Numerical analyses and results

In this section, we performed several numerical analyses to demonstrate the relevance of the proposed models mentioned above. First and foremost, we performed Monte Carlo simulations to examine the difference in pricing between a constant-volatility and a stochastic-volatility Asian option. Pricing results under both volatility conditions as well as computer operation time have been compared. Following that, we compared the approximate analytical solutions for the stochastic-volatility standard Asian option and SVFSAO with the benchmark of Monte Carlo simulation solutions. The outcomes obtained from this investigation reveal how well-suited the approximate analytical solutions are in terms of speed. The results specific to the accuracy of these high frequency tools will be discussed later on.

Table 4.1: Comparison of Standard Asian Options with stochastic and constant volatilities.

		(1)	**(2)**	**(3)**	**(4)**	**(5)**
μ	ξ	**MC(SV)**	**MC(CV)**	**Relative Error [(2) - (1)]/(1)**	**Time (SV)**	**Time (CV)**
0.0	0.3	6.060499 (0.0017)	6.076221 (0.0018)	0.259430%	167.615163	82.914033
	0.6	6.010829 (0.0021)	6.076221 (0.0018)	1.087907%	171.201748	82.914033
	0.9	5.935575 (0.0022)	6.076221 (0.0018)	2.369547%	171.234190	82.914033
0.1	0.3	6.143191 (0.0021)	6.076221 (0.0018)	-1.090139%	166.428761	82.914033
	0.6	6.096965 (0.0023)	6.076221 (0.0018)	-0.340221%	174.662555	82.914033
	0.9	6.018788 (0.0023)	6.076221 (0.0018)	0.954239%	196.584983	82.914033
0.2	0.3	6.233629 (0.0020)	6.076221 (0.0018)	-2.525129%	187.597290	82.914033
	0.6	6.185505 (0.0025)	6.076221 (0.0018)	-1.766776%	182.708622	82.914033
	0.9	6.100589 (0.0021)	6.076221 (0.0018)	-0.399429%	187.051886	82.914033

Notes: The parameters are set as follows: $S_0 = 100$, $r = 10\%$, $d = 0\%$, $\sigma_0 = 30\%$, and $T = 180$ days. We assume the option has an arithmetic average strike price. Option prices in column (1) are derived by applying a Monte Carlo simulation method with 100,000 paths repeated 50 times under stochastic volatility conditions. Options prices in column (2) are derived by applying a similar Monte Carlo simulation under constant volatility conditions. For columns (1) and (2), standard errors are in parentheses. Column (3) indicates the relative error between the standard Asian call options with constant volatility and the benchmark of standard Asian options with stochastic volatility. Columns (4) and (5) respectively report the time required for simulation under stochastic and constant volatility conditions. See Notebooks *Monte Carlo simulation with CV* and *Monte Carlo simulation with SV* for details.

4.2.1 Effect of stochastic volatility on standard and forward-starting Asian options

Since the stochastic volatility characteristics of financial assets are extensively accepted, Lin and Chang (2020) assert that the valuation of Asian options with constant volatility frequently entails model hazards. Consequently, we used the Monte Carlo simulation in various situations to draw attention to the differences in pricing between an Asian option under the circumstances of constant volatility and stochastic volatility. As part of these simulations, it was assumed that there are 365 days in a year to guarantee consistency with reality (Hsu et al., 2020).

Table 4.2: Comparison of Forward-Starting Asian Options with stochastic and constant volatilities.

		(1)	**(2)**	**(3)**	**(4)**	**(5)**
μ	ξ	**MC(SV)**	**MC(CV)**	**Relative Error [(2) - (1)]/(1)**	**Time (SV)**	**Time (CV)**
0.0	0.3	4.023445 (0.0017)	4.038512 (0.0013)	0.374479%	212.652013	75.119881
	0.6	3.979053 (0.0017)	4.038512 (0.0013)	1.494319%	186.669047	75.119881
	0.9	3.905389 (0.0016)	4.038512 (0.0013)	3.408698%	188.135205	75.119881
0.1	0.3	4.099035 (0.0016)	4.038512 (0.0013)	-1.476509%	182.392460	75.119881
	0.6	4.050894 (0.0018)	4.038512 (0.0013)	-0.305648%	204.453134	75.119881
	0.9	3.977258 (0.0015)	4.038512 (0.0013)	1.540124%	184.744173	75.119881
0.2	0.3	4.169501 (0.0017)	4.038512 (0.0013)	-3.141593%	185.076957	75.119881
	0.6	4.123486 (0.0014)	4.038512 (0.0013)	-2.060734%	186.931598	75.119881
	0.9	4.052662 (0.0013)	4.038512 (0.0013)	-0.349152%	179.295950	75.119881

Notes: The parameters are set as follows: $S_0 = 100$, $r = 10\%$, $d = 0\%$, $\sigma_0 = 30\%$, $T = 180$ days and $A = 90$ days. We assume the option has an arithmetic average strike price. Option prices in column (1) are derived by applying a Monte Carlo simulation method with 100,000 paths repeated 50 times under stochastic volatility conditions. Options prices in column (2) are derived by applying a similar Monte Carlo simulation under constant volatility conditions. For columns (1) and (2), standard errors are in parentheses. Column (3) indicates the relative error between the forward-starting Asian call options with constant volatility and the benchmark of forward-starting Asian options with stochastic volatility. Columns (4) and (5) respectively report the time required for simulation under stochastic and constant volatility conditions. See Notebooks *Monte Carlo simulation with CV* and *Monte Carlo simulation with SV* for details.

Tables 4.1 and 4.2 illustrate the pricing results under stochastic and constant volatility conditions. Table 4.1 shows the values for a standard Asian call option while Table 4.2 presents the values for a forward-starting Asian call option. In both situations, as the drift of the asset variance μ rises, the option values increase in price. It is easily noticed that pricing differences between both volatility conditions are obvious, exceeding 2% for quite a few conditions, for both standard and forward-starting Asian call options. Such a relative error indicates that the Asian option pricing model under constant volatility may lead to mispricing (Hsu et al., 2020).

Moreover, the value of the two Asian options under both volatility circumstances increases as the maturity lengthens. As a result, the discrepancies in the option values grow as μ and T increase. Complete and more detailed tables are available in Appendices D and E.

Additionally, when the volatility of asset variance ξ goes up, we observe that the values of both Asian options with stochastic volatility are not always higher than those of options with constant volatility. Hence, ξ is negatively correlated with the option price. However, it can be determined that ξ has substantial impacts on the value of standard and forward-starting Asian options.

Ultimately, we have seen previously that speed and execution time are essential in the context of HFT. From this perspective, computation time is a critical component for HFTs as they monitor computer-based trading. On the one hand, it can be observed that the time required for calculating the Asian option price with a Monte Carlo method is around 82 seconds for the case of a constant-volatility standard Asian option and at least 166 seconds for the case of a standard Asian options with stochastic volatility. On the other hand, about 75 seconds are needed to compute the value of a constant-volatility FSAO (CVFSAO) whereas we need at least 182 seconds for the case of a SVFSAO. This plainly confirms that simulation-based pricing methods are too time-consuming and are not suitable for HFT (Hsu et al., 2020). In Chapter 1, we indeed pointed out that HFT players are competing for extreme speed.

The results provided in this analysis support the need to develop an approximate analytical solution which incorporates a dynamic process related to stochastic volatility with the aim of pricing Asian options much faster (Lin and Chang, 2020).

4.2.2 Performance of the approximate analytic solutions

The pricing performance of the approximate analytical standard path-dependent option solution is presented in Table 4.3. Table 4.4 exhibits the pricing efficiency of the approximate analytical solution for a SVFSAO.

In each case, we performed a Monte Carlo simulation with 100,000 paths and 50 repetitions under stochastic volatility conditions. We also computed the price of a stochastic-volatility standard and forward-starting Asian option under multiple scenarios of T. For a large sample path, the model proposed by Lin and Chang (2020) outperformed our Monte Carlo simulation in terms of pricing accuracy and computation time, or speed. However, the model put forward by Hsu et al. (2020) does not meet the same precision criterion. The computation time criterion is still satisfied.

In greater detail, we find the approximate analytical solutions in column 2 of both tables. By comparing those solutions to the Monte Carlo simulation under stochastic volatility conditions,

Table 4.3: Comparison of the approximate analytical solutions for a stochastic-volatility standard Asian option and the Monte Carlo method.

	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
T (days)	**MC(SV)**	**AAS**	**Relative error [(2) - (1)]/(1)**	**Time (MC)**	**Time (AAS)**	**Efficiency (4)/(5)**
30	2.17081 (0.0007)	2.191470	0.951707%	46.966152	9.947E-04	4.722E+04
60	3.201403 (0.0011)	3.225304	0.746560%	61.027743	9.975E-04	6.118E+04
90	4.035907 (0.0013)	4.069093	0.822274%	91.183405	9.968E-04	9.147E+04
120	4.774032 (0.0015)	4.814170	0.840746%	117.173805	9.978E-04	1.174E+05
180	6.066506 (0.0017)	6.131736	1.075248%	345.735595	9.983E-04	3.463E+05
240	7.22615 (0.0027)	7.305780	1.101972%	462.337343	9.975E-04	4.635E+05
365	9.360496 (0.0033)	9.476320	1.237363%	559.903618	9.971E-04	5.616E+05

Notes: The parameters are set as follows: $S_0 = 100$, $r = 10\%$, $d = 0\%$, $\mu = 0$, $\xi = 0.15$, $\sigma_0 = 30\%$. We assume the option has an arithmetic average strike price. Option prices in column (1) are derived by applying a Monte Carlo simulation method with 100,000 paths repeated 50 times under stochastic volatility conditions. The standard error is presented in parentheses. Options prices in column (2) are obtained using the analytical solution proposed in this paper. Column (3) indicates the relative error between the approximate analytical solutions and the benchmark Asian options with stochastic volatility. Columns (4) and (5) respectively report the time required for simulation using the Monte Carlo method and the approximate analytical solution. Column (6) presents the efficiency. See Notebooks *Monte Carlo simulation with SV* and *AAS for SAOs* for details.

we observe that the model proposed by Lin and Chang (2020) is tremendously close to the theoretical value as all relative errors are pretty close to 1%. Regarding SVFSAOs, the analytic solution proposed by Hsu et al. (2020) displays relative errors higher than 2%, except when the option's maturity is fixed at one month. Additionally, the accuracy of the developed model strongly deteriorates as the option's maturity increases.

Furthermore, it can be noted that the approximate analytical solution can be computed remarkably rapidly. Computation time is in reality more than thirty-five thousand times faster than the Monte Carlo solution, for any value of T. From Tables 4.3 and 4.4, it can be observed that the approximate analytic solution takes just about 9.9×10^{-4} seconds, or about less than one millisecond, for either a stochastic-volatility standard Asian call option or a SVFSAO. In contrast, a Monte Carlo simulation requires more than 46 seconds in the case of a standard Asian option and more than 35 seconds in the case of a FSAO. These findings support the effectiveness in terms of speed of the approximate analytical solutions in comparison to simulation-based techniques (Lin and Chang, 2020).

Table 4.4: Comparison of the approximate analytical solutions for a SVFSAO and the Monte Carlo method.

	(1)	(2)	(3)	(4)	(5)	(6)
T (days)	**MC(SV)**	**AAS**	**Relative error [(2) - (1)]/(1)**	**Time (MC)**	**Time (AAS)**	**Efficiency**
30	1.48137 (0.0005)	1.479515	-0.125217%	35.236479	9.971E-04	3.534E+04
60	2.170568 (0.0007)	2.083512	-4.010762%	62.954860	9.978E-04	6.309E+04
90	2.721173 (0.0010)	2.540900	-6.624820%	103.651845	9.968E-04	1.040E+05
120	3.19993 (0.0014)	2.921369	-8.705238%	128.741992	9.975E-04	1.291E+05
180	4.036593 (0.0015)	3.546827	-12.133174%	193.156189	9.961E-04	1.939E+05
240	4.771437 (0.0015)	4.059285	-14.925319%	282.866452	9.956E-04	2.841E+05
365	6.126815 (0.0019)	4.911774	-19.831530%	393.843780	9.971E-04	3.950E+05

Notes: The parameters are set as follows: $S_0 = 100$, $r = 10\%$, $d = 0\%$, $\mu = 0$, $\xi = 0.15$, $\sigma_0 = 30\%$, and $A = T/2$ trading days. We assume the option has an arithmetic average strike price. Option prices in column (1) are derived by applying a Monte Carlo simulation method with 100,000 paths repeated 50 times under stochastic volatility conditions. The standard error is presented in parentheses. Options prices in column (2) are obtained using the analytical solution proposed in this paper. Column (3) indicates the relative error between the approximate analytical solutions and the benchmark Asian options with stochastic volatility. Columns (4) and (5) respectively report the time required for simulation using the Monte Carlo method and the approximate analytical solution. Column (6) presents the efficiency. See Notebooks *Monte Carlo simulation with SV* and *AAS for FSAOs* for details.

In addition to that, simulations for average option valuation require a longer calculation time as the maturity increases. Whatever the maturity of the option, proposed approximate analytic solutions need constant calculation time to provide an appropriate price. Because of that, when the maturity of the option T increases, the proposed solutions even reach computational efficiency more than five hundred thousand times faster than the Monte Carlo simulation in the case of a standard Asian call option. In terms of FSAOs, computing speeds are nearly four hundred thousand times faster than that of a numerical method, which definitely strengthens the dominance of the developed solutions.

In short, based on the model developed by Hull and White (1987), both studies introduced in this chapter derive an approximate analytic solution for pricing Asian options with stochastic volatility. Indeed, a stochastic-volatility Asian option pricing model is required, as proven by numerical experiments in Subsection 4.2.1, because the constant-volatility Asian option model produces significant model errors, higher than one penny in most instances.

On the one hand, the valuation solution provided by Lin and Chang (2020) shows how accurate and effective it is when compared to large-sample simulation trials as the benchmark. On the other hand, the approximate analytic solution developed by Hsu et al. (2020) does not meet a similar accuracy regarding SVFSAOs. However, this second alternative delivers comparable satisfactory results in terms of speed. Numerical analyses thus reveal that the approximate analytical solutions do not lead to the conclusion of a common superior performance while they undeniably outperformed computing efficiency in comparison with the Monte Carlo approach.

Additionally, it becomes increasingly obvious that both approximate analytic solutions surpass simulation-based methods in terms of CPU time as maturity lengthens. To some extent, those models are actually straightforward to implement. Thanks to a noteworthy calculation speed, which could be further improved by using another computer language, the advanced pricing models could be interesting from a high-frequency perspective.

General Conclusion

Throughout this master thesis, we covered many aspects of market finance. This paper's objective was to propose of pricing solution for Asian options from a high-frequency perspective. This solution should allow HFTs to work in the context of extreme speed inherent to HFT.

To provide a viable solution, we set up the following methodology. At first sight, we used the academic website Google Scholar on which we collected various scientific papers and articles. Additionally, we downloaded several reports from some federal regulatory instances such as the SEC or the CFTC. In a second stage, we used the Anaconda Navigator to code some algorithms in Python.

In order to find a possible answer, we first reviewed the literature on HFT. This first chapter allowed us to set the context in which this master thesis has been written. The aim was to understand the high-frequency perspective from which we would work in the subsequent chapters. Therefore, we established a framework for option pricing. Next, we explored the functioning of option derivatives in general. Also, we saw that there are three main option styles, namely European, American, and Asian. In order to remain consistent with this master thesis' objective, we were more particularly interested in Asian options, or average options. In the third chapter then, we investigated multiple option pricing approaches. The main purpose was to set up a baseline for some following experimentation. To wrap this third chapter up, we applied those option pricing techniques to the practical case of META. We thus put into practice the observed theory through various numerical applications. Eventually, we provided two approximate analytic solutions for pricing Asian options in a high-frequency environment. Both standard and forward-starting average options have been scrutinised in numerical analyses. For this purpose, we compared the analytical solutions to a benchmark of Monte Carlo solutions.

The results of the numerical analyses carried out in the last chapter are mixed. On the one hand, the approximate analytic solution for a standard Asian option produces satisfying outcomes. This solution is in fact accurate and efficient. All relative errors with the benchmark are fairly small, around 1%. Furthermore, the developed solution is much more efficient in terms of computational speed since it provides a value in less than one millisecond compared to a few minutes for the Monte Carlo simulation. On the other hand, the developed approximate analytic solution for a forward-starting Asian option does not provide such interesting results. Actually, the solution's accuracy is far from being perfect. We observed considerable discrepancies with the Monte Carlo reference. The computational speed is however satisfied, with computation times similar to the previous solution. Although extremely faster than simulation-based methods, the computation times obtained can nonetheless be considered insufficiently effective from a high-frequency perspective. Consequently, these results unfortunately do not allow us to conclude that the two approximate analytical solutions are useful tools for HFTs.

The answer to the research question is therefore not obvious. Although we have succeeded to price Asian options using approximate analytic solutions, prices are not accurate in all cases. Moreover, the context of HFT is not fully respected because the calculation times are still too long from a high-frequency standpoint.

Also, a few limitations have hindered the writing of this master thesis. First of all, I had no experience with the Anaconda Navigator and its software. Therefore, I had to get used to the technical use of these environments, which took me some time to adapt. On top of that, I have to admit that my skills in coding were somewhat limited as I had no real experience in that field. Although I have taken some basic training courses on coding in Python during my internship, I did not have sufficient skills to manage such a large project with such complex algorithms. Learning the Python syntax and all its features also took some time. From a practical point of view, it can be added that my aging computer often struggled to handle multiple Jupyter Notebooks and Excel files at once, which sometimes slowed down the writing of this master thesis. Primarily, the time required for some computations was longer. As a matter of example, the Monte Carlo method for Asian options displayed an error when the number of simulations was one million so that I could not refine the calculation accuracy for a fixed strike Asian option on META. Ultimately, although it is relatively straightforward to learn, the Python programming language is not the most powerful language in terms of computational time. This is a fundamental limitation given the high-frequency context in which we have been working. Indeed, we argue that a computation time of about one millisecond may not be fast enough from a high-frequency perspective, even if it is considerably decreased compared to simulation-based techniques. A feasible solution to this issue could be to code in another computer language. We have seen in this master thesis that Java, C# or C++ are common programming language for algorithms.

For a future research, we can thus think of doing similar paper in a more powerful programming language. In a high-frequency perspective, we will then lean towards C++, or possibly Java. This would be done with the aim of obtaining much high calculation speeds, of the order of a few microseconds, or ideally nanoseconds, which would match the characteristics of HFT. Moreover, we could consider developing an approximate analytic solution for fixed strike Asian options, as opposed to floating strike which we have tackled in this master thesis. At last, we could incorporate the Heston volatility model into the pricing of Asian options. Using a similar Monte Carlo simulation, we could compare the pricing differences with the Hull and White's volatility model.

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